Math 6130 Notes. Fall 2002.

7. Basic Maps. Recall from §3 that a regular map of affine varieties is "the same" as a homomorphism of coordinate rings (going the other way). Here, we look at how algebraic properties of the homomorphisms relate to the geometry of three basic types of regular maps of varieties.

Definition: A regular map $\Phi : X \to Y$ of quasi-projective varieties is a *closed embedding* if Φ is an isomorphism from X to a closed subset $Z \subset Y$.

Examples: (a) A quasi-projective variety Y is affine (resp. projective), by definition, if there is a closed embedding $\Phi : Y \to \mathbf{C}^n$ (resp. $\Phi : Y \to \mathbf{CP}^n$).

(b) Let $\Phi : X \to Y$ be any regular map. The *graph* of Φ is the regular map $\gamma_\Phi := (\mathrm{id}_X, \Phi) : X \to X \times Y$, whose image is $\Gamma_\Phi := \{(x, \Phi(x)) \mid x \in X\}$. This is a closed subset of $X \times Y$, by Proposition 6.5(b), since it is equal to the inverse image of the diagonal under the regular map:

$$(\Phi \circ \pi_X, \pi_Y) : X \times Y \to Y \times Y$$

Moreover, the compositions: $X \stackrel{\gamma_\Phi}{\to} \Gamma_\Phi \stackrel{\pi_X}{\to} X$ and $\Gamma_\Phi \stackrel{\pi_X}{\to} X \stackrel{\gamma_\Phi}{\to} \Gamma_\Phi$ are clearly the identity maps, so γ_Φ is an isomorphism from X to Γ_Φ. That is, the graph γ_Φ is a closed embedding, and so **every** regular map Φ factors:

$$\Phi = \pi_Y \circ \gamma_\Phi : X \to X \times Y \to Y$$

as a closed embedding (the graph) followed by a projection.

Proposition 7.1: A regular map of affine varieties $\Phi : X \to Y$ is a closed embedding if and only if $\Phi^* : \mathbf{C}[Y] \to \mathbf{C}[X]$ is *surjective.*

Proof: If Φ is an isomorphism from X to a closed subvariety $Z \subset Y$, then $\mathbf{C}[Z] = \mathbf{C}[Y]/I(Z)$, where $I(Z)$ is the (prime) ideal of regular functions on Y that vanish on Z, so

$$\Phi^* : \mathbf{C}[Y] \to \mathbf{C}[Y]/I(Z) \stackrel{\sim}{\to} \mathbf{C}[X]$$

is surjective. On the other hand, if $\Phi : X \to Y$ is any regular map of affine varieties, and if $\Phi^* : \mathbf{C}[Y] \to \mathbf{C}[X]$ is surjective, let $I = \ker(\Phi^*)$ and consider $Z = V(I) \subset Y$. If $y \notin Z$, let $\overline{g} \in I$ be a regular function such that $\overline{g}(y) \neq 0$. Then y cannot be in the image of Φ, because if it were, say $\Phi(x) = y$, then $\Phi^*(\overline{g})(x) = \overline{g}(y) \neq 0$, and so $\overline{g} \notin \ker(\Phi^*)$. Thus Φ factors through a map $\Phi : X \to Z \subset Y$, and the induced $\Phi^* : \mathbf{C}[Z] \to \mathbf{C}[X]$ is an isomorphism, so Φ is an isomorphism from X to Z. That is, $\Phi : X \to Y$ is a closed embedding.

So surjective homomorphisms of coordinate rings $\Phi^* : \mathbf{C}[Y] \to \mathbf{C}[X]$ correspond to closed embeddings of affine varieties. It makes sense next to ask about the *injective* homomorphisms of coordinate rings.

Definition: A regular map $\Phi : X \to Y$ of affine varieties is *dominant* if the map on coordinate rings $\Phi^* : \mathbf{C}[Y] \to \mathbf{C}[X]$ is injective.

Examples: (a) (Projections) Recall that $\mathbf{C}[X \times Y] \cong \mathbf{C}[X] \otimes_{\mathbf{C}} \mathbf{C}[Y]$ for affine varieties (Exercise 6.2). It follows that the two projections:

$$\pi_X : X \times Y \to X \quad \text{and} \quad \pi_Y : X \times Y \to Y$$

are dominant, with corresponding injective maps:

$$\pi_X^* : \mathbf{C}[X] \to \mathbf{C}[X \times Y]; g \mapsto g \otimes 1 \text{ and } \pi_Y^* : \mathbf{C}[Y] \to \mathbf{C}[X \times Y]; h \mapsto 1 \otimes h$$

(b) (Open embeddings) Recall that a basic open set $U = X - V(\overline{g}) \subset X$ in a affine variety X is again an affine variety (Proposition 3.6) and that its affine coordinate ring is (Exercise 3.3(a)):

$$\mathbf{C}[X]_{\overline{g}} := \left\{ \frac{\overline{f}}{\overline{g}^m} \mid \overline{f} \in \mathbf{C}[X], m \geq 0 \right\} \subset \mathbf{C}(X)$$

The inclusion mapping $\Phi : U \to X$ therefore gives the homomorphism:

$$\Phi^* : \mathbf{C}[X] \to \mathbf{C}[U] = \mathbf{C}[X]_{\overline{g}}; \quad \overline{f} \mapsto \frac{\overline{f}}{1}$$

which is evidently injective. So the open embedding of U is a dominant map.

(c) (Blow Up) The regular map $\Phi : \mathbf{C}^2 \to \mathbf{C}^2$ given by:

$$\Phi(x, y) = (x, xy)$$

is dominant, since $\Phi^* : \mathbf{C}[s, t] \to \mathbf{C}[x, y]$ is given by $\Phi^*(s) = x, \Phi^*(t) = xy$, and there are no polynomial relations between x and xy. Notice that Φ induces an isomorphism of open sets:

$$\Phi : \mathbf{C}^2 - V(x) \xrightarrow{\sim} \mathbf{C}^2 - V(s)$$

but that Φ "blows down" the y-axis $V(x)$ to the origin $(0, 0) \in \mathbf{C}^2$. We will see in §8 that the image of Φ is not isomorphic to a quasi-projective variety.

Proposition 7.2: $\Phi : X \to Y$ is dominant if and only if $\Phi(X) \subset Y$ is dense.

Proof: Suppose Φ is dominant, and $U = Y - V(\overline{g})$ is a basic open set. Since Φ^* is injective and $\overline{g} \neq 0$ (always for basic open sets) then $\Phi^*(\overline{g}) \neq 0$, which is to say that for some $x \in X$, $\Phi^*(\overline{g})(x) = \overline{g}(\Phi(x)) \neq 0$. But this means that $\Phi(x) \in U$, and so $U \cap \Phi(X) \neq \emptyset$. Since this is true of all basic open sets, it follows that $\Phi(X)$ is dense.

On the other hand, if $\Phi(X)$ is dense then $\Phi(X)$ intersects each basic open set $U = Y - V(\overline{g})$, and so for each $\overline{g} \neq 0$, we can find an $x \in X$ so that $\Phi(x) = y$ and $\overline{g}(y) \neq 0$, which implies that $\Phi^*(\overline{g})(x) = \overline{g}(y) \neq 0$, so $\Phi^*(\overline{g}) \neq 0$. Thus Φ is dominant.

Following the previous proposition, it makes sense to define in general:

Definition: A regular map $\Phi : X \to Y$ of quasi-projective varieties X, Y is *dominant* if $\Phi(X)$ is dense in Y.

Observation: If X is a *projective* variety, then the factorization of Φ:

$$\Phi = \pi_Y \circ \gamma_\Phi : X \to X \times Y \to Y$$

as a composition of a closed embedding followed by a projection shows that Φ is a *closed map*, i.e. it takes closed sets to closed sets, since both the closed embedding γ_Φ and the projection map π_Y (by Proposition 6.7(b)) are closed. Since the only subset of Y that is both dense and closed is Y, we see that:

Corollary 7.3: A dominant map of *projective* varieties is always surjective.

It is time to begin to pay more attention to the *fields of rational functions* of a quasi-projective variety. Recall that when $Y \subset \mathbf{C}^n$ is affine, then:

$$\mathbf{C}(Y) = (\text{the field of fractions of } \mathbf{C}[Y])$$

and when X is projective, then $\mathbf{C}(X) \subset$ (the field of fractions of $\mathbf{C}[X]$) is the set of fractions that are a ratio of regular functions of the same degree.

If $Y \subset X$ is a quasi-projective variety (open in the projective variety X), we defined $\mathcal{O}_Y(U)$ to be the elements of $\mathbf{C}(X)$ that are regular on U. Since *any* rational function on X is regular on some open set $U \subset X$, and then also on the (non-empty!) intersection $V = U \cap Y \subset Y$, it follows that:

$$\mathbf{C}(X) = \bigcup_{U \subset X} \mathcal{O}_X(U) = \bigcup_{V \subset Y} \mathcal{O}_Y(V)$$

and this will be our *definition* of $\mathbf{C}(Y)$, so that in particular $\mathbf{C}(X) = \mathbf{C}(Y)$.

Proposition 7.4: A dominant map $\Phi : X \to Y$ of quasi-projective varieties induces a well-defined (injective!) pull-back of fields of rational functions:

$$\Phi^* : \mathbf{C}(Y) \hookrightarrow \mathbf{C}(X)$$

Proof: Recall that by definition, Φ satisfies: $\Phi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\Phi^{-1}(U))$ for all open subsets $U \subset Y$. If $\phi \in \mathbf{C}(Y)$ is nonzero, then the domain of ϕ is some such open set U, and *because* Φ *is dominant*, the inverse image $\Phi^{-1}(U)$ is non-empty. Thus $\Phi^*(\phi)$ is a nonzero(!) rational function on X, and the pull-back is defined on all rational functions (and injective).

Remarks: (a) When Φ is not dominant, the pull-back is similarly defined, but only for rational functions $\phi \in \mathbf{C}(Y)$ whose domains intersect $\Phi(X)$.

(b) When X and Y are affine, then $\Phi^* : \mathbf{C}(Y) \hookrightarrow \mathbf{C}(X)$ is the extension of the injection $\Phi^* : \mathbf{C}[Y] \hookrightarrow \mathbf{C}[X]$ to an injection of the fields of fractions.

We can reverse Proposition 7.4, in the spirit of Proposition 3.5, but only if we allow for dominant *rational* maps.

Proposition 7.5: For quasi-projective varieties X and Y, each inclusion of fields $\iota : \mathbf{C}(Y) \hookrightarrow \mathbf{C}(X)$ gives rise to a rational map:

$$\Phi : X - -> Y$$

such that the regular map $\Phi|_U : U \to Y$ (from the domain of Φ) is dominant and the two inclusions of fields agree, i.e. $\Phi^* = \iota$.

Proof: If $X \subset \overline{X} \subset \mathbf{CP}^m$ and $Y \subset \overline{Y} \subset \mathbf{CP}^n$, construct the rational map Φ out of the rational functions $\iota(\frac{\overline{y}_i}{\overline{y}_j}) \in \mathbf{C}(X)$ as in Proposition 4.4.

Definition: A rational map $\Phi : X - -> Y$ is *birational* if it is dominant (thought of as a regular map from its domain) and Φ^* is a field isomorphism.

Examples: Examples (b) and (c) (the open embedding and blow-up) of dominant maps are also birational. Note that neither is an isomorphism!

We will have more to say about birational maps in §9. For now, I want to concentrate on another very important class of regular maps:

Definition: A regular map $\Phi : X \to Y$ of affine varieties is *finite* if the pull-back $\Phi^* : \mathbf{C}[Y] \to \mathbf{C}[X]$ makes $\mathbf{C}[X]$ a finitely generated $\mathbf{C}[Y]$-*module*.

Examples: (a) A closed embedding $\Phi : X \to Y$ is finite, since in that case Φ^* is surjective (Proposition 7.1), so $\mathbf{C}[X]$ is generated by 1 as a $\mathbf{C}[Y]$-module.

(b) The open embedding $Y - V(\overline{g}) =: U \hookrightarrow Y$ of a basic open set in an affine variety is **not** finite because, as a module, $\mathbf{C}[U] = \mathbf{C}[X]_{\overline{g}}$ is freely generated by $1, \overline{g}^{-1}, \overline{g}^{-2}, \overline{g}^{-3}, \ldots$

(c) A projection $\pi_X : X \times Y \to X$ is only finite if Y is a point.

(d) Let $X = V(y^2 - x) \subset \mathbf{C}^2$ be the parabola (on its side) and let $\Phi : X \to \mathbf{C}^1$ be the projection to the x-axis, which is two-to-one except at the origin. Then the homomorphism of coordinate rings is the inclusion:

$$\Phi^* : \mathbf{C}[x] \hookrightarrow \mathbf{C}[X] = \mathbf{C}[x, y]/\langle y^2 - x \rangle$$

so Φ is dominant. It is finite since $\mathbf{C}[X]$ is generated by $1, \overline{y}$ as a $\mathbf{C}[x]$-module.

(e) (Affine Noether Normalization) If X is any affine variety, then the Noether Normalization Lemma of §1 showed that there is a polynomial ring $\mathbf{C}[y_1, \ldots, y_d] \subset \mathbf{C}[X]$ so that $\mathbf{C}[X]$ is a finitely generated $\mathbf{C}[y_1, \ldots, y_d]$-module. So the associated regular map $\Phi : X \to \mathbf{C}^d$ is both dominant and finite.

Observation: If $\Phi : X \to Y$ is finite and dominant, then the inclusion of fields $\Phi^* : \mathbf{C}(Y) \hookrightarrow \mathbf{C}(X)$ makes $\mathbf{C}(X)$ into a finite field extension of $\mathbf{C}(Y)$. The generators of $\mathbf{C}[Y]$ as a $\mathbf{C}[X]$-module span $\mathbf{C}(Y)$ as a $\mathbf{C}(X)$-vector space.

Proposition 7.5: A finite map $\Phi : X \to Y$ of affine varieties always has the following geometric properties:

(a) the "fibers" $\Phi^{-1}(y)$ are finite sets for all $y \in Y$.

(b) Φ is a closed map (i.e. takes closed sets to closed sets)

Remark: Each of these requires a new lemma from commutative algebra, so we will preceed each part of the proof with the relevant algebra result.

Chinese Remainder Lemma: If A is a commutative ring with 1 and if $m_1, \ldots, m_r \subset A$ are distinct maximal ideals, then the natural map:

$$A/m_1 \cap \ldots \cap m_r \to \times_{i=1}^r A/m_i$$

is an isomorphism. In particular, if A is a $\mathbf{C}$-algebra of dimension r as a vector space over $\mathbf{C}$, then A has at most r distinct maximal ideals.

Proof: We proceed by induction on r. If $r = 1$, the result is a tautology. Suppose the Lemma holds for r distinct maximal ideals. Since there are exactly r maximal ideals in the product of fields: $\times_{i=1}^{r} A/m_i$ it follows that $m_1, ..., m_r$ must be the only maximal ideals containing $m_1 \cap .. \cap m_r$. Thus if m_{r+1} is a new maximal ideal, then $m_{r+1} + (m_1 \cap ... \cap m_r) = A$, and by the usual Chinese Remainder Theorem, we get the desired isomorphism:

$$A/m_1 \cap ... \cap m_r \cap m_{r+1} \xrightarrow{\sim} \times_{i=1}^{r} A/m_i \times A/m_{r+1}.$$

Remark: There may be fewer maximal ideals than the rank of A in the lemma! For example, $A = \mathbf{C}[x]/\langle x^r \rangle$ has rank r as a vector space over $\mathbf{C}$, but only one maximal ideal, namely $\langle \overline{x} \rangle$. This is no contradiction to the lemma, of course, which is a tautology in this case.

Proof of (a): If $\Phi : X \to Y$ is any regular map and $Z \subset Y$ is a closed set, then we always have $\Phi^{-1}(Z) = V(\langle \Phi^* I(Z) \rangle)$. That is because a point $a \in X$ satisfies $\Phi(a) \in Z$ if and only if for each regular function $\overline{g} \in I(Z)$ the pull-back $\Phi^*\overline{g}$ vanishes at a. Thus we need to show that if Φ is a finite map, then $V(\langle \Phi^* m_b \rangle)$ is a finite set whenever m_b is the maximal ideal of functions vanishing at a point $b \in Y$. To see this, consider the following diagram:

$$\begin{array}{ccc} \mathbf{C}[Y] & \stackrel{\Phi^*}{\to} & \mathbf{C}[X] \\ \downarrow & & \downarrow \\ \mathbf{C} = \mathbf{C}[Y]/m_b & \stackrel{\overline{\Phi}^*}{\to} & A = \mathbf{C}[X]/\langle \Phi^* m_b \rangle \end{array}$$

Now either $A = 0$, in which case $\langle \Phi^* m_b \rangle = \mathbf{C}[X]$, and there are **no** points in $\Phi^{-1}(b)$, or else $A \neq 0$, in which case A is a $\mathbf{C}$-algebra and the generators $\overline{f}_1, ..., \overline{f}_r$ of $\mathbf{C}[X]$ as a $\mathbf{C}[Y]$-module also span A as a $\mathbf{C}$-vector space. Since points of $\Phi^{-1}(b)$ correspond to the maximal ideals of A by the Nullstellensatz, it follows from the Chinese Remainder Lemma that $\Phi^{-1}(b)$ in fact consists of at most r points.

Example: Consider again the example of the sideways parabola. Then $\Phi^{-1}(b) = V(\langle \overline{x} - b \rangle) = V(\langle \overline{y}^2 - b \rangle) \subset X$, and $\mathbf{C}[X] = \mathbf{C}[x, y]/\langle y^2 - x \rangle \cong \mathbf{C}[y]$. Thus there are two possibilities:

$$A \cong \mathbf{C}[y]/\langle y^2 - b \rangle \text{ has } \begin{cases} 2 \text{ maximal ideals if } b \neq 0 \text{ and} \\ \\ 1 \text{ maximal ideal if } b = 0 \end{cases}$$

corresponding to the 2 square roots of b when $b \neq 0$ and the 1 when $b = 0$.

Localization: A subset $S \subset A$ of a commutative ring with 1 is *multiplicative* if $1 \in S, 0 \notin S$ and S is closed under multiplication. If A is a domain, the *localization* of A at a multiplicative subset $S \subset A$ is the domain:

$$A \subset A_S := \left\{ \frac{f}{s} \mid f \in A, s \in S \right\} \subset K$$

sitting between A and its field of fractions K (which is A_S for $S = A - \{0\}$).

Examples: (a) The set $S = A - P$ is multiplicative when P is a prime ideal. In this case, we always write A_P instead of A_S (by convention).

(b) The set $S = \{1, f, f^2, f^3, ...\} \subset A$ is multiplicative if $f^n \neq 0$ for all n, and in this case, we always write A_f instead of A_S (also by convention).

Note: We've already used this notation to write $\mathbf{C}[U] = \mathbf{C}[X]_{\overline{g}}$ when $U = X - V(\overline{g})$ is a basic open subset of an affine variety X.

A Fundamental Correspondence: The prime ideals $Q \subset A_S$ are in a bijection with the prime ideals $P \subseteq A - S$ (i.e. $P \subset A$ and $P \cap S = \emptyset$) via:

$$Q \mapsto Q \cap A \text{ and } P \mapsto P_S = \left\{ \frac{b}{s} \mid b \in P, s \in S \right\} = PA_S$$

and $P \subseteq P' \Leftrightarrow P_S \subseteq P'_S$ under this bijection.

In particular, the domain A_P is always a *local ring*, meaning that it has a unique maximal ideal, i.e. the ideal $m \subset A_P$ such that $m \cap A = P = A - S$ includes all the other prime ideals of the form $Q \cap A$, so each $Q \subseteq m$.

The correspondence is valid for arbitrary ideals $J \subset A_S$ and $I \subset A - S$:

$$J \mapsto J \cap A \text{ and } I \mapsto I_S = IA_S$$

but it is not a bijection, because although $J \mapsto J \cap A \mapsto (J \cap A)_S = J$ always, it can happen (when I isn't prime) that $I \mapsto I_S \mapsto I_S \cap A$ and $I \neq I_S \cap A$. In other words, there can be *more* ideals in $A - S$ than in A_S. This, and the fact that the correspondence preserves inclusions allows us to conclude that:

If A is Noetherian, than each localization A_S is also Noetherian.

Remark: This correspondence is a counterpart to the fundamental bijection between the ideals $J/I \subset A/I$ and the ideals $J \subset A$ that contain I.

Cohen-Seidenberg Going-Up Lemma: If $B \subset A$ are Noetherian domains and A is finitely generated as a B-module, then every prime ideal $P \subset B$ is equal to $Q \cap B$ for some prime ideal $Q \subset A$.

Proof: Given $P \subset B$, let $S = B - P$ and localize *both* A and B with respect to the multiplicative set $S \subset B \subset A$, to get a diagram of inclusions:

$$\begin{array}{ccc} A & \subset & A_S \\ \cup & & \cup \\ B & \subset & B_P \end{array}$$

Since the unique maximal ideal $m \subset B_P$ satisfies $m \cap B = P$ by the fundamental correspondence, once we find a prime ideal $n \subset A_S$ such that $n \cap B_P = m$, then $(n \cap A) \cap B = m \cap B = P$, and then we take $Q = n \cap A$.

But let $n \subset A_S$ be any *maximal* ideal. I claim that $B_P/(B_P \cap n)$ is a field, hence that $B_P \cap n = m$, as desired. To see this, notice that A_S/n is a field which is finitely generated as a $B_P/(B_P \cap n)$-module (by the images of the generators of A as a B-module). If $\beta \in B_P/(B_P \cap n)$ were nonzero with no inverse, then the chain of submodules:

$$B_P/(B_P \cap n) \subset \beta^{-1} B_P/(B_P \cap n) \subset \beta^{-2} B_P/(B_P \cap n) \subset ... \subset A_S/n$$

would never be stationary, violating the Noetherian property of $B_P/(B_P \cap n)$. (We already saw this argument in the proof of the Nullstellensatz!)

Proof of 7.5 (b): Let $Z = \overline{\Phi(X)}$. Then $\Phi : X \to Z$ is a dominant map of affine varieties by Proposition 7.2 (and it is also finite). If Φ is closed as a map to Z, then it is closed as a map to Y, so we may assume Φ is dominant, and hence that $\Phi^* : \mathbf{C}[Y] \hookrightarrow \mathbf{C}[X]$ is an inclusion of rings. It also suffices to prove that Φ takes *irreducible* closed sets to (irreducible) closed sets.

Given an irreducible closed set $V(Q) \subseteq X$, let $P = \Phi^{*-1}(Q) \subset \mathbf{C}[Y]$. It is always true that $\Phi(V(Q)) \subseteq V(P)$ since if $a \in V(Q)$ and $\overline{f} \in P$, then $\overline{f}(\Phi(a)) = \Phi^*(\overline{f})(a) = 0$. We claim that finiteness gives us $\Phi(V(Q)) = V(P)$.

Suppose $b \in V(P)$ and let $P \subseteq m_b \subset \mathbf{C}[Y]$ be the corresponding maximal ideal. We want to find a prime ideal $Q \subseteq n \subset \mathbf{C}[X]$ with $\Phi^{*-1}(n) = m_b$, because then from the previous paragraph $\Phi(V(n)) \subseteq V(m_b) = b$. But the Going-Up Lemma says that since $\mathbf{C}[X]/Q$ is a finitely generated $\mathbf{C}[Y]/P$-module, there is a prime ideal $n/Q \subset \mathbf{C}[X]/Q$ such that $\overline{\Phi}^{*-1}(n/Q) = m_b/P$, and then the prime ideal $n \subset \mathbf{C}[X]$ has the desired property.

Warning: Unlike the case with closed embeddings and dominant maps, there is no simple geometric criterion that is equivalent to finiteness. The converse to Proposition 7.5, for example, is false, as seen with the following.

Example: Let $U = X - \{(1,1)\}$ where $X = V(y^2 - x) \subset \mathbf{C}^2$ is the parabola. Then the projection $\pi : U \to \mathbf{C}^1$ is still a closed map with finite fibers, but $\mathbf{C}[U] = \mathbf{C}[X]_{\overline{y-1}}$ is not finitely generated as a $\mathbf{C}[x]$-module (or $\mathbf{C}[X]$-module).

So instead of a simple geometric generalization of finiteness to maps of varieties, we have to define finiteness in terms of an affine cover:

Definition: A regular map $\Phi : X \to Y$ of quasi-projective varieties is *finite* if there is an open cover $Y = \cup U_i$ of Y by affine varieties such that each $V_i := \Phi^{-1}(U_i)$ is also affine, and such that the maps $\Phi|_{V_i} : V_i \to U_i$ are finite.

Remark: It is not true in general that the preimage of an open affine variety is affine. For a simple example, consider the inclusion $\Phi : \mathbf{C}^n - \{0\} \hookrightarrow \mathbf{C}^n$. You proved in Exercise 3.4 that $\mathbf{C}^n - \{0\} = \Phi^{-1}(\mathbf{C}^n)$ is not affine.

A priori the definition of finiteness looks as though it depends upon the choice of an affine cover of Y. This is not the case, but it requires some work to prove independence of the cover, via the following:

Proposition 7.6: If $\Phi : X \to Y$ is a finite dominant map of quasi-projective varieties and if $U \subset Y$ is *any* open, affine subset, then $\Phi^{-1}(U)$ is also affine, and the induced map $\Phi|_{\Phi^{-1}(U)} : \Phi^{-1}(U) \to U$ is a finite map of affine varieties.

Proof: First, we will prove that $\Phi^{-1}(U)$ is affine using the following:

Criterion for Affineness: If f is a regular function on a quasi-projective variety W, let $V(f) = \{w \in W \mid f(w) = 0\}$, and let $U = W - V(f)$ be the quasi-projective generalization of a basic open set of an affine variety. Now suppose there are regular functions $f_1, ..., f_k$ on W such that the sets $U_i := W - V(f_i)$ are affine varieties and suppose that $\sum_{i=1}^k f_i g_i = 1$ for some other regular functions $g_1, ..., g_k$ on W. Then W is itself an affine variety.

Proof of the Criterion: Let each $\mathbf{C}[U_i] = \mathbf{C}[x_{i1}, ..., x_{im_i}]/P_i$ and let $A = \mathcal{O}_W(W)$. We first need to know that A is of the form $\mathbf{C}[x_1, ..., x_m]/P$, as it needs to be if W is affine and $A = \mathbf{C}[W]$. By definition of a regular function,

$$\mathbf{C}[U_1] \cap ... \cap \mathbf{C}[U_k] = A \subset \mathbf{C}(W)$$

Notice that $U_i \cap U_j = U_j - V(\overline{f}_i) \subset U_j$, so each $U_i \cap U_j$ is affine, and $\mathbf{C}[U_i \cap U_j] = \mathbf{C}[U_j]_{\overline{f}_i}$. Thus if $a \in \mathbf{C}[U_i]$ is arbitrary, then $af_i^{n_j} \in \mathbf{C}[U_j]$ for each U_j and some n_j, and so if $n = \max\{n_j\}$, then $af_i^n \in A$. That is:

$$\mathbf{C}[U_i] = A_{f_i} \text{ for each } U_i$$

In particular, each of the $\overline{x}_{il} \in \mathbf{C}[U_i]$ satisfies $\overline{x}_{il} f_i^n \in A$ for some fixed n chosen large enough (to work for all $\overline{x}_{il}$ and all i at once). If $a \in A$, then there are polynomials $p_1, ..., p_k$ in the generators of each $\mathbf{C}[U_i]$ such that:

$$a = p_i(\overline{x}_{i1}, ..., \overline{x}_{im_i}) \in \mathbf{C}[U_i]$$

and we can find an N such that the right side of each expression:

$$af_i^N = f_i^N p_i(\overline{x}_{i1}, ..., \overline{x}_{im_i})$$

is a polynomial in the $\overline{x}_{il} f_i^n$ (for the fixed n) and f_i. Take:

$$(\sum_{i=1}^{k} f_i g_i)^{(N-1)k+1} = 1 \quad \text{giving} \quad (\sum_{i=1}^{k} f_i g_i)^{(N-1)k+1} a = a$$

Each term on the left has some af_i^N in it, so we see that a (on the right) is a polynomial in the fixed elements $(\overline{x}_{il} f_i^n), f_i, g_i \in A$. That is, as desired:

$$A = \mathbf{C}[x_1, ..., x_m]/P = \mathbf{C}[y_{il}, s_i, t_i]/P; \quad y_{il} \mapsto \overline{x}_{il} f_i^n, s_i \mapsto f_i, t_i \mapsto g_i$$

Next, let $X \subset \mathbf{C}^m$ be the affine variety associated to A, and consider:

$$\Psi : W \to X; \quad w \mapsto (\Psi^*(x_1)(w), ..., \Psi^*(x_m)(w))$$

Then $\Psi^* : \mathbf{C}[X] \xrightarrow{\sim} \mathcal{O}_W(W)$ is an isomorphism (both are isomorphic to A) and if $V_i = X - V(f_i)$ are basic open affine sets of X corresponding to the f_i, then the V_i cover X because $\sum f_i g_i = 1$, and each $\Psi|_{U_i} : U_i \to V_i$ is an isomorphism because both varieties are affine, and $\Psi|_{U_i}^* : \mathbf{C}[V_i] \xrightarrow{\sim} \mathbf{C}[U_i]$ (both are isomorphic to A_{f_i}). But now it follows that Ψ is an isomorphism, with inverse given locally by the $\Psi|_{U_i}^{-1}$ maps!

This finishes the proof of the Criterion for Affineness.

Back to the Proposition: Let $Y = \cup U_i$ be an affine cover over which Φ is finite, i.e. each $\Phi|_{\Phi^{-1}(U_i)} : \Phi^{-1}(U_i) \to U_i$ is a finite map of affine varieties. If $U_i - V(f) \subset U_i$ is a basic open set, then Φ is finite over $U_i - V(f)$, since each $\Phi^{-1}(U_i - V(f)) = \Phi^{-1}(U_i) - V(f)$, and the generators for $\mathbf{C}[\Phi^{-1}(U_i)]$ as a $\mathbf{C}[U_i]$-module also generate $\mathbf{C}[\Phi^{-1}(U_i)]_f$ as a $\mathbf{C}[U_i]_f$-module.

This means that an arbitrary affine $U \subset Y$ is covered by basic open sets (of the U_i) over which Φ is finite. But these in turn are covered by basic open sets of U (and of themselves!) over which Φ is finite. That is, there is a cover $U = \cup_{i=1}^k (U - V(f_i))$ over which Φ is finite, and $\sum_{i=1}^k f_i g_i = 1$ by the Nullstellensatz. But now the cover $\Phi^{-1}(U) = \cup_{i=1}^k (\Phi^{-1}(U - V(f_i))$ by generalized basic open sets $\Phi^{-1}(U) - V(f_i)$ satisfies the criterion for affineness!

Moreover, if $a_{i1}, ..., a_{im_i}$ generate each $\mathbf{C}[\Phi^{-1}(U)]_{f_i}$ as a $\mathbf{C}[U]_{f_i}$-module, then there is an n so that each $a_{il} f_i^n \in \mathbf{C}[\Phi^{-1}(U)]$, and I claim that all the $a_{il} f_i^n$ together generate $\mathbf{C}[\Phi^{-1}(U)]$ as a $\mathbf{C}[U]$-module. Indeed, the f_i and g_i in the expression $\sum f_i g_i = 1$ are all in $\mathbf{C}[U]$, so as before we can take a large enough N, and consider $(\sum f_i g_i)^N a = a$ to see that the claim is true.

Example: (Projective Noether Normalization) If $X \subset \mathbf{CP}^n$ is a projective variety, then the y_i in the inclusion $\mathbf{C}[y_1, ..., y_d] \subset \mathbf{C}[X]$ making $\mathbf{C}[X]$ a finite $\mathbf{C}[y_1, ..., y_d]$-module all have degree 1, giving a rational map:

$$\Phi : X -\!\!> \mathbf{CP}^{d-1};\ a \mapsto (y_1(a); ...; y_d(a))$$

which I claim is actually a regular map. If $a \in X$, then there is some coordinate $\overline{x}_i \in \mathbf{C}[X]$ such that $\overline{x}_i(a) \neq 0$. But then some power $\overline{x}_i^k$ is a linear combination of lower powers of the $\overline{x}_i$ with coefficients in $\mathbf{C}[\underline{y}] = \mathbf{C}[y_1, ..., y_d]$ (otherwise $\mathbf{C}[\underline{y}] \subset \overline{x}_i \mathbf{C}[\underline{y}] \subset \overline{x}_i^2 \mathbf{C}[\underline{y}] \subset ... \subset \mathbf{C}[X]$ would never be stationary) and then it follows that some $y_j(a) \neq 0$ and Φ is regular.

Φ is also a finite map. If we cover $\mathbf{CP}^n$ by affines $U_j = \mathbf{CP}^n - V(y_j)$, then each $V_j = X - V(y_j) = \Phi^{-1}(U_j)$ is also affine by Exercise 4.4 and if $\overline{F}_1, ..., \overline{F}_m \in \mathbf{C}[X]$ are (homogeneous) generators of $\mathbf{C}[X]$ as a $\mathbf{C}[\underline{y}]$-module, then $\frac{\overline{F}_1}{y_j^{d_1}}, ..., \frac{\overline{F}_m}{y_j^{d_m}} \in \mathbf{C}[V_j]$ generate $\mathbf{C}[V_j]$ as a $\mathbf{C}[U_j]$-module.

Remark: It follows immediately from the definition that a finite map of quasi-projective varieties also has properties (a) and (b) of Proposition 7.5. That is, it is a closed map with finite fibers. We've already seen that the converse is not true, but a deep theorem of Grothendieck says that the converse only fails to be true "in the obvious way."

Theorem (Grothendieck): If $\Phi : X \to Y$ is a regular map of varieties with finite fibers, then there is a finite map:

$$\Phi' : X' \to Y$$

such that $X \subset X'$ is an open subset, and $\Phi = \Phi'|_X : X \to Y$.

In other words, every map with finite fibers (closed or not) is a finite map minus a closed subset of the domain. This theorem is very powerful, but its proof is outside the scope of this course (it is something to look forward to!) Here is one simple consequence:

Corollary 7.7: A map $\Phi : X \to Y$ of *projective* varieties with finite fibers is always a finite map. (We've already seen that such a map is always closed!)

Proof: By Grothendieck's Theorem, Φ is the restriction of a finite map from a variety containing X as an open subset. But projective varieties are proper, so X only sits inside *itself* as an open set!

We will see another important application of the theorem in §9.

Exercises 7.

6. Generalize the construction of Projective Noether Normalization to show:

If $X \subset \mathbf{CP}^m$ and $Y \subset \mathbf{CP}^n$ are projective varieties, then an injective graded homomorphism of homogeneous coordinate rings $\phi : \mathbf{C}[X] \hookrightarrow \mathbf{C}[Y]$ (i.e. satisfying $\phi(\mathbf{C}[X]_d) \subset \mathbf{C}[Y]_d$) gives rise to a rational map:

$$\Phi : Y - \!\!> X$$

and if $\mathbf{C}[Y]$ is a finitely generated $\mathbf{C}[X]$-module, then Φ is regular and finite.