Math 6130 Notes. Fall 2002.

2. Another Hilbert Theorem. When we think about projective geometry, we need to regard the polynomial ring as a graded object:

$$\mathbf{C}[x_0, x_1, ..., x_n] = \bigoplus_{d=0}^{\infty} \mathbf{C}[x_0, x_1, ..., x_n]_d$$

decomposing an arbitrary polynomial into a (finite) sum of homogeneous polynomials (i.e. sums of monomials of the same degree), so we get:

$$\dim(\mathbf{C}[x_0, x_1, ..., x_n]_d) = \binom{n+d}{n} = \#\{\text{monomials of degree } d \text{ in } x_0, ..., x_n\}$$

An ideal $I \subset \mathbf{C}[x_0, x_1, ..., x_n]$ is *homogeneous* if it, too decomposes:

$$I = \bigoplus_{d=0}^{\infty} I_d = \bigoplus_{d=0}^{\infty} I \cap \mathbf{C}[x_0, x_1, ..., x_n]_d$$

and then by the Hilbert Basis Theorem, such an ideal satisfies:

$$I = \langle F_1, ..., F_m \rangle = \{\sum_{i=1}^{m} g_i F_i \mid g_1, ..., g_m \in \mathbf{C}[x_0, x_1, ..., x_n]\}$$

for homogeneous polynomials $F_1, ..., F_m$ (usually not all of the same degree).

More generally, a module M over $\mathbf{C}[x_0, x_1, ..., x_n]$ is *graded* if:

$$M = \bigoplus_{d \in \mathbf{Z}} M_d$$

as a sum of complex vector spaces, such that the multiplication maps send: $\mathbf{C}[x_0, x_1, ..., x_n]_d \times M_e \to M_{d+e}$. A homomorphism $\phi : M \to N$ of graded $\mathbf{C}[x_0, x_1, ..., x_n]$-modules is a *graded homomorphism* if each $\phi(M_d) \subseteq N_d$.

Examples: (a) A graded module M can be twisted to yield another graded module:

$$M(e) := \bigoplus_{d \in \mathbf{Z}} M_{d+e}$$

so that, for instance, if we regard $S = \mathbf{C}[x_0, x_1, ..., x_n]$ as a graded module over itself, then we obtain the modules:

$$S(e) = \bigoplus_{d=-e}^{\infty} \mathbf{C}[x_0, x_1, ..., x_n]_{d+e}$$

(b) A homogeneous $F \in \mathbf{C}[x_0, x_1, ..., x_n]_e$ yields graded homomorphisms:

$$M(-e) \to M; \quad m \mapsto Fm$$

In particular, the graded homomorphism:

$$S(-e) \to S; \quad g \mapsto Fg$$

is an isomorphism onto the ideal $\langle F \rangle \subset S$. The generators of a homogeneous ideal $I = \langle F_1, ..., F_m \rangle$ determine a graded homomorphism of graded modules:

$$\bigoplus_{i=1}^{m} S(-e_i) \to S; \quad (g_1, ..., g_m) \mapsto \sum_{i=1}^{m} F_i g_i$$

whose image is I, and whose kernel is the "graded module of relations."

(c) The kernel, cokernel and image of a graded homomorphism are graded.

Definition: If the dimensions $\dim(M_d)$ are all finite, then:

$$h_M(d) := \dim(M_d)$$

is the *Hilbert function* of the graded module M.

Hilbert's Polynomial Growth Theorem: If M is a finitely generated graded $\mathbf{C}[x_0, x_1, ..., x_n]$-module, then the dimensions $\dim(M_d)$ are all finite, and there is a d_0 (depending upon M) and a *polynomial* $H_M(d)$ such that:

$$h_M(d) = H_M(d) \text{ for all } d \geq d_0$$

Proof: There is a natural basis for the free abelian group of polynomial functions $P : \mathbf{Z} \to \mathbf{Z}$. Namely,

$$\left\{1, \binom{d}{1}, \binom{d}{2}, \binom{d}{3}, ...\right\}$$

with the pleasant property, noticed by Pascal, that if:

$$P(d) = a_0 + a_1 \binom{d}{1} + ... + a_m \binom{d}{m}$$

then

$$P(d+1) - P(d) = a_1 + a_2 \binom{d}{1} + ... + a_m \binom{d}{m-1}$$

We prove the theorem by induction on the number of variables in the polynomial ring $\mathbf{C}[x_0, x_1, ..., x_n]$, noting that the Hilbert function of a finite dimensional vector space V over $\mathbf{C}$ is 0 in large degrees, so $H_V(d) = 0$.

Suppose $n \geq 0$ and consider the exact sequence:

$$(*) : \quad 0 \to K \to M \stackrel{x_n}{\to} M(1) \to N(1) \to 0$$

where the map in the middle is the map from Example (b) applied to the module $M(1)$ (and the polynomial x_n) and K and $N(1)$ are the (graded!) kernel and cokernel, respectively. Multiplication by x_n acts trivially on K and $N(1)$, so they are (finitely generated) graded modules over the ring $\mathbf{C}[x_0, x_1, ..., x_n]/\langle x_n \rangle \cong \mathbf{C}[x_0, ..., x_{n-1}]$, and we are ready to apply induction.

Namely, the Hilbert functions are additive on exact sequences, so:

$$h_M(d+1) - h_M(d) = h_K(d) - h_{N(1)}(d)$$

and thus by induction $h_M(d)$ is either always infinite or always finite. But for sufficiently small d (i.e. smaller than the degrees of all the generators) $h_M(d) = 0$. So $h_M(d)$ is always finite. Next, if d_0 is chosen so $h_K(d) = H_K(d)$ and $h_{N(1)}(d) = H_{N(1)}(d)$ are polynomial functions for $d \geq d_0$, then their difference is a polynomial, so:

$$h_M(d+1) - h_M(d) = a_1 + a_2 \binom{d}{1} + ... + a_m \binom{d}{m-1}$$

for some integers $a_1, ..., a_m$ and all $d \geq d_0$. Setting $a_0 = h_M(d_0) - \sum a_i \binom{d_0}{i}$ then gives:

$$h_M(d) = H_M(d) = a_0 + a_1 \binom{d}{1} + ... + a_m \binom{d}{m}$$

for all $d \geq d_0$, as desired.

Definition: $H_M(d)$ is the *Hilbert polynomial* of the graded module M.

Observation: Hilbert polynomials, like Hilbert functions, are additive on exact sequences of graded modules.

Examples: (a) The Hilbert polynomial of $S = \mathbf{C}[x_0, x_1, ..., x_n]$ itself is:

$$H_S(d) = \binom{d+n}{n} = \frac{1}{n!}d^n + \text{ lower order}$$

and we can take d_0 as small as $-n$ since $0 = \binom{0}{n} = \binom{1}{n} = ... = \binom{n-1}{n}$.

(b) The Hilbert polynomial of the quotient:

$$0 \to \langle F \rangle \to S \to S/\langle F \rangle \to 0$$

by a principal homogeneous ideal generated by F of degree e is:

$$H_{S/\langle F \rangle}(d) = \binom{d+n}{n} - \binom{d-e+n}{n} = \frac{e}{(n-1)!}d^{n-1} + \text{ lower order}$$

valid for $d_0 \geq -n+e$.

Before we leave graded rings, I want to consider their homogeneous ideals:

Definition: The unique maximal homogeneous ideal:

$$\langle x_0, ..., x_n \rangle \subset \mathbf{C}[x_0, x_1, ..., x_n]$$

is called the *irrelevant* maximal ideal. It contains every homogeneous ideal.

The Projective Hilbert Nullstellensatz: The homogeneous prime ideals $P \subset \mathbf{C}[x_0, x_1, ..., x_n]$ that are maximal with the property of being properly contained in the irrelevant maximal ideal are all of the form:

$$\langle y_1, ..., y_n \rangle \subset \langle x_0, ..., x_n \rangle \subset \mathbf{C}[x_0, x_1, ..., x_n]$$

where the $y_i = \sum_{j=0}^{n} a_{ij}x_j$ are independent linear forms.

Proof: Such ideals are evidently prime and maximal (in this sense). To see that they are the only ones, consider the ordinary Nullstellensatz. More precisely, if $P \subset \langle x_0, ..., x_n \rangle$ is any homogeneous prime ideal properly contained in the irrelevant maximal ideal, then $V(P)$ contains the origin and at least one other point $p \in \mathbf{C}^{n+1}$. Otherwise, by Corollary 1.4, we'd have a contradiction with $P = I(V(P)) = \langle x_0, ..., x_n \rangle$. Once a homogeneous ideal I satisfies $p \in V(I) \subset \mathbf{C}^{n+1}$, then $V(I)$ must contain the entire line $\mathbf{C}p = \{\lambda p \mid \lambda \in \mathbf{C}\}$, and then I must be contained in the ideal $I(\mathbf{C}p)$, which is already of the form $\langle y_1, ..., y_n \rangle$ where the y_i are any n independent linear forms whose common solution set is the line $\mathbf{C}p$. So $P = \langle y_1, ..., y_n \rangle$.

Note: The maximal ideals are thus precisely the homogeneous prime ideals in $\mathbf{C}[x_0, x_1, ..., x_n]$ such that $V(I) \subset \mathbf{C}^{n+1}$ is a single line through the origin. Recall that the ordinary maximal ideals in $\mathbf{C}[x_1, ..., x_n]$ are precisely the ordinary prime ideals such that $V(I) \subset \mathbf{C}^n$ is a single point.

Definition: *Complex projective space* $\mathbf{CP}^n$ is the set of lines through the origin in $\mathbf{C}^{n+1}$. That is, it is the set of equivalence classes:

$$\left\{\mathbf{C}^{n+1} - 0\right\} / \sim \text{ where } p \sim \lambda p \text{ for } \lambda \in \mathbf{C}^*$$

and if $0 \neq p = (p_0, ..., p_n)$, then the equivalence class containing p is denoted:

$$(p_0 : p_1 : ... : p_n) \in \mathbf{CP}^n$$

Remarks: (a) $\mathbf{CP}^n$ is a union $\mathbf{C}^n \cup \mathbf{CP}^{n-1}$ of:

$$\mathbf{C}^n = \{(p_1, ..., p_n)\} = \{(1 : p_1 : ... : p_n)\} \text{ and}$$
$$\mathbf{CP}^{n-1} = \{(0 : p_1 : ... : p_n)\}$$

since the first coordinate is either non-zero or zero, and if it is non-zero, then it can be set to 1 (in the equivalence class) and the other coordinates are then fixed. Geometrically, this means that we should think of $\mathbf{CP}^n$ as being "ordinary" $\mathbf{C}^n$ with $\mathbf{CP}^{n-1}$ giving us the extra "points at infinity" which we identify with the slopes of the lines through the origin in $\mathbf{C}^n$. We can, of course, continue this process to get a "stratification:"

$$\mathbf{CP}^n = \mathbf{C}^n \cup \mathbf{C}^{n-1} \cup ... \cup \mathbf{C}^1 \cup \mathbf{C}^0$$

by successive points at infinity.

(b) As in the proof of the Nullstellensatz above, it makes sense to say that $(p_0 : ... : p_n) \in V(I)$ or $(p_0 : ... : p_n) \notin V(I)$ for a homogeneous ideal I, since this property does not depend upon the representative of $(p_0 : ... : p_n)$.

Corollary 2.1: Given homogeneous $F_1, ..., F_m \in \mathbf{C}[x_0, x_1, ..., x_n]$, then either there is a point $(p_0 : ... : p_n) \in \mathbf{CP}^n$ so that $F_i(p_0 : ... : p_n) = 0$ for all i or else there is an N so that:

$$x_j^N = \sum_{i=1}^n G_{ij} F_i \text{ can be solved with homogeneous } G_{ij} \in \mathbf{C}[x_0, x_1, ..., x_n]$$

(and finding the G_{ij} is hard, of course)

Proof: If there is no such point, then $\langle F_1, ..., F_m \rangle$ does not belong to any of the homogeneous maximal prime ideals, by the Projective Nullstellensatz, so it follows that $V(\langle F_1, ..., F_m \rangle) = \{0\} \in \mathbf{C}^{n+1}$. That is, by Corollary 1.4:

$$\sqrt{\langle F_1, ..., F_n \rangle} = \langle x_0, ..., x_n \rangle$$

so that $x_i^{N_i} \in \langle F_1, ..., F_m \rangle$, and then we let N be the maximum of the N_i.

Finally, consider the following two processes:

Homogenizing: Instead of $x_1, ..., x_n$, let n variables be labelled $\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}$. Then a (non-homogeneous) $f \in \mathbf{C}[\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}]$ of degree d *homogenizes* to $h(f) := x_0^d f \in \mathbf{C}[x_0, x_1, ..., x_n]_d$. More generally, an ideal $I \subset \mathbf{C}[\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}]$ homogenizes to:

$$h(I) := \langle h(f) \mid f \in I \rangle \subset \mathbf{C}[x_0, x_1, ..., x_n]$$

and we know that finitely many of the $h(f_i)$ will suffice to generate $h(I)$.

The geometric significance of this process is as follows. If

$$V(I) \subset \mathbf{C}^n$$

is the algebraic set associated to I, then homogenizing produces:

$$V(h(I)) \subset \mathbf{CP}^n$$

with the property that $V(h(I)) \cap \mathbf{C}^n = V(I)$. In other words, homogenizing the ideal tells us how to add points at infinity to an algebraic set in $\mathbf{C}^n$ in order to get an algebraic set in $\mathbf{CP}^n$.

Dehomogenizing: A homogeneous $I \subset \mathbf{C}[x_0, x_1, ..., x_n]$ *dehomogenizes* to:

$$d_0(I) := \left\{ F\left(1, \frac{x_1}{x_0}, ..., \frac{x_n}{x_0}\right) \mid F \in I \right\} \subset \mathbf{C}\left[\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}\right]$$

(with respect to x_0) which is already an ideal. Geometric Interpretation: the intersection of the algebraic set $V(I) \subset \mathbf{CP}^n$ with $\mathbf{C}^n$ is $V(d_0(I)) \subset \mathbf{C}^n$.

These operations are nearly inverses. For all ideals $I \subset \mathbf{C}[\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}]$:

$$d_0(h(I)) = I$$

(Geometry: when we add points at infinity, we don't add extra finite ones.) For homogeneous prime ideals $P \subset \mathbf{C}[x_0, x_1, ..., x_n]$ not containing x_0:

$$h(d_0(P)) = P$$

and in general, $I \subseteq h(d_0(I))$. (Geometry: If we intersect such a $V(P)$ with $\mathbf{C}^n$ and then add points at infinity, we get $V(P)$ back. Otherwise we may lose some of the points at infinity by this process.)

Example (The Twisted Cubic): Consider the set:

$$V := \left\{ \left(\frac{t}{s}, \left(\frac{t}{s}\right)^2, \left(\frac{t}{s}\right)^3 \right) \mid \frac{t}{s} \in \mathbf{C} \right\} \subset \mathbf{C}^3$$

(the *affine twisted cubic*) and its "one point compactification:"

$$\overline{V} := \{(s^3 : s^2t : st^2 : t^3) \mid (s:t) \in \mathbf{CP}^1\} = V \cup \{(1:0:0:0)\} \subset \mathbf{CP}^3$$

(the "projective twisted cubic"). It is easy to see that (in variables $\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}$):

$$I(V) = I := \left\langle \frac{x_2}{x_0} - \left(\frac{x_1}{x_0}\right)^2, \frac{x_3}{x_0} - \left(\frac{x_1}{x_0}\right)\left(\frac{x_2}{x_0}\right) \right\rangle$$

since, for example I is the kernel of the homomorphism:

$$\mathbf{C}\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right] \to \mathbf{C}\left[\frac{t}{s}\right];\ \frac{x_1}{x_0} \mapsto \frac{t}{s},\ \frac{x_2}{x_0} \mapsto \left(\frac{t}{s}\right)^2,\ \frac{x_3}{x_0} \mapsto \left(\frac{t}{s}\right)^3$$

and $V = V(I)$. When we homogenize this ideal, we do **not** get:

$$J = \langle x_2x_0 - x_1^2, x_3x_0 - x_1x_2 \rangle$$

because, for example, $\left(\frac{x_2}{x_0}\right)^2 - \left(\frac{x_1}{x_0}\right)\left(\frac{x_3}{x_0}\right) \in I$ but $x_2^2 - x_1x_3 \notin J$ since it is not a linear combination of $x_2x_0 - x_1^2$ and $x_3x_0 - x_1x_2$. So we do not homogenize an ideal in general just by homogenizing its generators. On the other hand,

$$\langle x_2x_0 - x_1^2, x_3x_0 - x_1x_2, x_2^2 - x_1x_3 \rangle$$

is prime, and is the kernel of the homomorphism:

$$\mathbf{C}[x_0, x_1, x_2, x_3] \to \mathbf{C}[s,t];\ x_0 \mapsto s^3,\ x_1 \mapsto s^2t,\ x_2 \mapsto st^2,\ x_3 \mapsto t^3$$

so this is the homogenized ideal, and the ideal of the projective twisted cubic. Moreover, from the homomorphism above:

$$\left(\mathbf{C}[x_0, x_1, x_2, x_3]/I(\overline{V})\right)_d \cong \mathbf{C}[s,t]_{3d}$$

so the Hilbert polynomial of $\mathbf{C}[x_0, x_1, x_2, x_3]/I(\overline{V})$ is:

$$H_{\mathbf{C}[x_0,x_1,x_2,x_3]/I(\overline{V})}(d) = H_{\mathbf{C}[s,t]}(3d) = 3d + 1$$

Exercises 2.

First, a little review. A sequence of homomorphisms of abelian groups:

$$(**)\ 0 \to A_0 \stackrel{\phi_1}{\to} A_1 \stackrel{\phi_2}{\to} \cdots \stackrel{\phi_n}{\to} A_n \to 0$$

is a *complex* if each $\phi_{i+1} \circ \phi_i = 0$, and it is *exact* if, in addition, each

$$\ker(\phi_{i+1}) = \mathrm{im}(\phi_i)$$

so that, in particular, ϕ_1 is injective, and ϕ_n is surjective.

1. Check the homological assertions of this section. Namely, check that:

(a) The image, kernel and cokernel of a graded homomorphism of graded $\mathbf{C}[x_0, x_1, ..., x_n]$-modules $\phi : M \to N$ are all graded modules.

(b) If $(**)$ above is an exact sequence of finite dimensional *vector spaces* V_i over $\mathbf{C}$ (with linear maps ϕ_i), then the dimensions of the V_i satisfy:

$$\sum_i (-1)^i \dim(V_i) = 0$$

(c) If $(**)$ above is an exact sequence of graded $\mathbf{C}[x_0, x_1, ..., x_n]$-modules M^i and graded homomorphisms $\phi^i : M^i \to M^{i+1}$ (I raised the subscript in this case so it won't be confused with the degree) then

$$\sum_i (-1)^i h_{M^i}(d) = 0 \text{ and } \sum_i (-1)^i H_{M^i}(d) = 0$$

(assuming that the Hilbert functions and Hilbert polynomials exist).

(d) If $F \in \mathbf{C}[x_0, x_1, ..., x_n]_e$ and M is a finitely generated graded module over $\mathbf{C}[x_0, x_1, ..., x_n]$, let $N = M/FM$. If the Hilbert polynomial of M is:

$$H_M(d) = \frac{a}{k!} d^k + \ \{\text{lower order terms}\}$$

show that $\deg(H_N(d)) \geq k - 1$ and that if $Fm \neq 0$ for all $m \neq 0$ in M, then:

$$H_N(d) = \frac{ea}{(k-1)!} d^{k-1} + \ \{\text{lower order terms}\}$$

so $H_N(d)$ has degree exactly $k - 1$ in this case.

2. Find generators for the homogeneous ideals $I(V)$ and Hilbert polynomials of $\mathbf{C}[x_0, x_1, ..., x_n]/I(V)$ for each of the following algebraic subsets $V \subset \mathbf{CP}^n$.

(a) $\{p_1, ..., p_m\} \subset \mathbf{CP}^n$, a set of (distinct) points.

(b) the pair of skew lines $\{(a : b : 0 : 0)\} \cup \{(0 : 0 : c : d)\} \subset \mathbf{CP}^3$.

(c) the pair of intersecting lines $\{(a : b : 0 : 0)\} \cup \{(0 : b : c : 0)\} \subset \mathbf{CP}^3$.

(d) the *rational normal curve* in $\mathbf{CP}^n$, i.e.

$$\{(s^n : s^{n-1}t : s^{n-2}t^2 : ... : t^n) \mid (s : t) \in \mathbf{CP}^1\}$$

(this is the natural generalization of the projective twisted cubic)

3. Suppose $F_1, ..., F_m \in \mathbf{C}[x_0, x_1, ..., x_n]$ are homogeneous of degrees $e_1, ..., e_m$.

(a) If $I = \langle F_1, ..., F_m \rangle$ and $m \leq n$, show that:

$$\deg(H_{\mathbf{C}[x_0,x_1,...,x_n]/I}(d)) \geq n - m$$

and if each F_{i+1} is not a zero divisor in $\mathbf{C}[x_0, x_1, ..., x_n]/\langle F_1, ..., F_i \rangle$ then:

$$\deg(H_{\mathbf{C}[x_0,x_1,...,x_n]/I}(d)) = \frac{\prod_{i=1}^m e_i}{(n-m)!} d^{n-m} + \{\text{lower order terms}\}$$

Ideals with generators with this property are *complete intersection ideals*.

(b) For the homogeneous ideals $I(V)$ in Exercise 2.2, show that:

(i) $I(V)$ is a complete intersection when V is the pair intersecting lines, but $I(V)$ is not a complete intersection when the lines are skew.

(ii) Prove that if $n \geq 3$, then the ideal of the rational normal curve in $\mathbf{CP}^n$ is not a complete intersection ideal.

(c) Show that $V(F_1) \cap ... \cap V(F_m) \neq \emptyset$ for any choice of homogeneous polynomials $F_1, ..., F_m$ in (a). (Hint: Use (a) and the Proj Nullstellensatz)

(d) If $F_1, ..., F_n$ generate a complete intersection ideal in $\mathbf{C}[x_0, x_1, ..., x_n]$, so that in particular, $\mathbf{C}[x_0, x_1, ..., x_n]/\langle F_1, ..., F_n \rangle = \prod_{i=1}^n e_i$, then show that $V(\langle F_1, ..., F_n \rangle) \subset \mathbf{CP}^n$ is a finite set, consisting of *at most* $\prod_{i=1}^n e_i$ points.

4. Prove the assertions in the text about homogenizing and dehomogenizing:

(a) Prove that for ideals $I \subset \mathbf{C}[\frac{x_1}{x_0}, ... \frac{x_n}{x_0}]$

$$d_0(h(I)) = I$$

(b) Prove that for homogeneous prime ideals $x_0 \notin P \subset \mathbf{C}[x_0, x_1, ..., x_n]$,

$$h(d_0(P)) = P$$

5. (a) For homogeneous ideals $I \subset \mathbf{C}[x_0, x_1, ..., x_n]$, prove that

$$\sqrt{I} = \{f \in \mathbf{C}[x_0, x_1, ..., x_n] \mid f^N \in I \text{ for some } N > 0\}$$

is also a homogeneous ideal.

(b) If $V \subseteq \mathbf{C}^{n+1}$ is a union of lines through the origin, prove that:

$$I(V) = \{f \in \mathbf{C}[x_0, x_1, ..., x_n] \mid f(a_0, a_1, ..., a_n) = 0 \ \forall (a_0, a_1, ..., a_n) \in V\}$$

is a homogeneous ideal.

(c) For $V \subseteq \mathbf{CP}^n$, let $I(V)$ be the homogeneous ideal in (b) for the union of lines in $\mathbf{C}^{n+1}$ parametrized by V. Prove the "projective" Corollary 1.4:

For homogeneous ideals $I \subset \mathbf{C}[x_0, x_1, ..., x_n]$, either $V(I) = \emptyset \in \mathbf{CP}^n$ or:

$$I(V(I)) = \sqrt{I}$$

(In particular, $I(V(P)) = P$ when $P \subset \langle x_0, ..., x_n \rangle$ is a homogeneous prime.)