Math 6130 Notes. Fall 2002.

5. Basic Examples. Before we go further with the theory, it is time to introduce some of the basic examples of affine and projective varieties.

d-uple Embeddings: These are the regular maps:

$$v_d : \mathbf{CP}^n \to \mathbf{CP}^{\binom{n+d}{n}-1}$$

obtained by selecting **all** the monomials of degree d

$$F_I = x_0^{i_0} \cdots x_n^{i_n} \in \mathbf{C}[x_0, x_1, ..., x_n]_d$$

as $I = (i_0, ..., i_n)$ ranges over multi-indices with $i_0 + ... + i_n = d$, sending

$$(a_0 : ... : a_n) \mapsto (\cdots : F_I(a_0 : ... : a_n) : \cdots)$$

for some ordering (usually "lexicographic") of the multi-indices I.

Example: The rational normal curves of Exercises 2 are the images:

$$v_d : \mathbf{CP}^1 \to \mathbf{CP}^d;\ (s : t) \mapsto (s^d : s^{d-1}t : ... : t^d)$$

of the d-uple embeddings of $\mathbf{CP}^1$.

Proposition 5.1: v_d is always an isomorphism onto the *Veronese variety*:

$$V_{n,d} := V(\langle x_I x_J - x_K x_L \mid I + J = K + L \rangle) \subset \mathbf{CP}^{\binom{n+d}{n}-1}$$

Proof: It is an easy check to see that the image of v_d is contained in $V_{n,d}$. Let $J = (j_0, ..., j_n)$ be any multi-index with $j_0 + ... + j_n = d - 1$ and consider the multi-indices $J_i := (j_0, ..., j_i + 1, ..., j_n)$. The projection:

$$\Psi : V_{n,d} -\!\!> \mathbf{CP}^n;\ (\cdots : a_I : \cdots) \mapsto (a_{J_0} : ... : a_{J_n})$$

is independent of the choice of J, since another J' and corresponding J'_i would satisfy $a_{J'_i} a_{J_j} = a_{J_i} a_{J'_j}$ for all i, j and any $(... : a_I : ...) \in V_{n,d}$. Thus:

$$(a_{J_0} : ... : a_{J_n}) = (a_{J'_0} : ... : a_{J'_n})$$

whenever both are defined. But Ψ is then a regular map inverting v_d.

Example (The Veronese Surface): This is the case $n = d = 2$.

Let x, y, z be the variables for $\mathbf{CP}^2$, and order the monomials:

$$x^2, xy, xz, y^2, yz, z^2$$

so the d-uple embedding is $v_2 : \mathbf{CP}^2 \to \mathbf{CP}^5$, with variables $x_0, ..., x_5$ and the equations of the Veronese surface $V_{2,2} \subset \mathbf{CP}^5$ are:

$$x_0x_3 - x_1^2, x_0x_4 - x_1x_2, x_0x_5 - x_2^2, x_1x_4 - x_2x_3, x_1x_5 - x_2x_4, x_3x_5 - x_4^2$$

and **all** are needed to generate $I(V_{2,2})$, since they are linearly independent!

The Veronese variety also has an important role in the theory:

Corollary 5.2: *Every* basic open subset $X_{\overline{F}} := X - V(\overline{F})$ of a projective variety $X \subset \mathbf{CP}^n$ is isomorphic to an affine variety.

Proof: The isomorphism $v_d : \mathbf{CP}^n \to V_{n,d}$ of Proposition 5.1 induces isomorphisms $v_d : Y \to Y_{n,d}$ from any (quasi)-projective variety $Y \subset \mathbf{CP}^n$ to its image $Y_{n,d} \subset V_{n,d}$ (Exercise 4.5). In particular, the image of X is a projective variety $X_{n,d}$, and if we decompose any homogeneous polynomial $F = \sum a_I F_I$ lifting $\overline{F} \in \mathbf{C}[x_0, x_1, ..., x_n]/I(X)$ as a sum of monomials of degree d, then the image of $X_{\overline{F}}$ is the basic open subset $X_{n,d} - V(\sum a_I x_I)$, which is isomorphic to an affine variety by Exercise 4.4 (c).

Corollary 5.3: Every quasi-projective variety is covered by affine varieties.

Proof: Let $Y \subset \overline{Y} \subset \mathbf{CP}^n$ be a quasi-projective variety in its closure, and let $Z = \overline{Y} - Y$ be the "boundary" of Y. Then Z is an algebraic set, which is therefore an intersection $Z = V(\overline{F}_1) \cap ... \cap V(\overline{F}_m)$ of hypersurfaces. Then $Y = \overline{Y}_{\overline{F}_1} \cup ... \cup \overline{Y}_{\overline{F}_m}$ is a union of the basic open sets which were shown to be affine varieties in Corollary 5.2.

Segre Embeddings: These are the varieties:

$$S_{m,n} \subset \mathbf{CP}^{(m+1)(n+1)-1}$$

where the coordinates of $\mathbf{CP}^{(m+1)(n+1)-1}$ are labelled x_{ij}, indexed by pairs $i \in \{0, ..., m\}$ and $j \in \{0, ..., n\}$ and the equations of $S_{m,n}$ are:

$$S_{m,n} = V(\langle x_{ij}x_{kl} = x_{il}x_{kj} \text{ for all } i, j, k, l\rangle)$$

Proposition 5.4: There is a natural bijection from $S_{m,n}$ to the Cartesian product $\mathbf{CP}^m \times \mathbf{CP}^n$ so that the (Cartesian) projections of sets:

$$\pi_m : S_{m,n} \to \mathbf{CP}^m \text{ and } \pi_n : S_{m,n} \to \mathbf{CP}^n$$

are regular maps, and if Y is any quasi-projective variety with regular maps:

$$p : Y \to \mathbf{CP}^m \text{ and } q : Y \to \mathbf{CP}^n$$

then the (unique!) resulting map $(p,q) : Y \to S_{m,n}$ is also regular.

Proof: The bijection is: $(\cdots : a_{ij} : \cdots) \mapsto ((a_{0j} : ... : a_{mj}), (a_{i0} : ... : a_{in}))$ if $a_{ij} \neq 0$. This is independent of the choice of i, j (using $a_{ij}a_{kl} = a_{il}a_{kj}$). The arrow is reversed by sending:

$$((a_0 : ... : a_m), (b_0 : ... : b_n)) \mapsto (\cdots : a_ib_j : \cdots)$$

Now given $Y \subset \mathbf{CP}^l$ and regular maps $p : Y \to \mathbf{CP}^m$ and $q : Y \to \mathbf{CP}^n$, then by Proposition 4.4, we can write p and q (in any neighborhood) as:

$$p = (F_0 : ... : F_m) \text{ and } q = (G_0 : ... : G_n)$$

and then $(p,q) = (\cdots : F_iG_j : \cdots)$ in that neighborhood, so it is regular.

The Segre embedding also has a role in the theory, allowing us to construct products of arbitrary quasi-projective varieties (see §6).

Example: (The Smooth Quadric Surface) Consider $m = n = 1$. Then there is only one non-redundant equation for $S_{1,1}$, namely:

$$S_{1,1} = V(x_{00}x_{11} - x_{01}x_{10}) \subset \mathbf{CP}^3$$

which (in $\mathbf{C}^4$) furnished our non-UFD example of a coordinate ring in §3. The two projections are (ordering the variables $x_{00}, x_{01}, x_{10}, x_{11}$):

$$\pi_1 : S_{1,1} \to \mathbf{CP}^1; (a : b : c : d) \mapsto (a : c) \text{ (or } (b : d)) \text{ and}$$

$$\pi_2 : S_{1,1} \to \mathbf{CP}^1; (a : b : c : d) \mapsto (a : b) \text{ (or } (c : d))$$

By Proposition 5.4, this quadric hypersurface is $\mathbf{CP}^1 \times \mathbf{CP}^1$ (as a set).

Affine Group Varieties: These are affine varieties that are also groups. To say precisely what it means for group multiplication to be a regular map, we will need products (in the next section), so for now we will be content to describe some of these varieties:

$\mathrm{GL}(n, \mathbf{C}) \subset \mathbf{C}^{n^2}$ (with coordinates x_{ij}) is the basic open set:

$$\mathrm{GL}(n, \mathbf{C}) = \mathbf{C}^{n^2} - \det(x_{ij})$$

consisting of the invertible matrices. This is isomorphic to an affine variety by Proposition 3.6. The inverse map is evidently regular:

$$i : \mathrm{GL}(n, \mathbf{C}) \to \mathrm{GL}(n, \mathbf{C});\ i(a_{ij}) = \det(a_{ij})^{-1}(A_{ij})$$

since the ijth minor A_{ij} of the matrix (a_{ij}) is a polynomial in the a_{ij}.

$\mathrm{SL}(n, \mathbf{C}) \subset \mathbf{C}^{n^2}$ is the irreducible closed subset $V(\det(x_{ij}) - 1) \subset \mathbf{C}^{n^2}$.

$\mathrm{PGL}(n+1, \mathbf{C}) = \mathbf{CP}^{(n+1)^2-1} - \det(x_{ij}) \subset \mathbf{CP}^{(n+1)^2-1}$ is the group of invertible matrices modulo scalars. It is an affine variety by Corollary 5.2. We have already seen this group in Exercise 4.3, where it was shown to be isomorphic to the group of automorphisms of $\mathbf{CP}^n$.

$\mathrm{O}(n, \mathbf{C}) \subset \mathrm{GL}(n, \mathbf{C})$ is the orthogonal group with equations (within $\mathbf{C}^{n^2}$):

$$(x_{ij})(x_{ij})^T = I_n$$

(more generally, $(x_{ij})S(x_{ij})^T = S$ for a non-degenerate symmetric matrix S).

Example: $\mathrm{O}(2, \mathbf{C})$ is isomorphic to two copies of $\mathbf{C}^*$. The equations are:

$$x_{11}^2 + x_{12}^2 = 1, x_{11}x_{21} + x_{12}x_{22} = 0, x_{21}^2 + x_{22}^2 = 1$$

for $\mathrm{O}(2, \mathbf{C}) \subset \mathbf{C}^4$. Thus:

$$\mathrm{O}(2, \mathbf{C}) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & -a_{11} \end{pmatrix} \text{ and } \begin{pmatrix} a_{11} & a_{12} \\ -a_{12} & a_{11} \end{pmatrix} \mid a_{11}^2 + a_{12}^2 = 1 \right\}$$

so both components are isomorphic to the affine variety $V(x^2+y^2-1) \subset \mathbf{C}^2$. But $x^2 + y^2 - 1 = (x+iy)(x-iy) - 1$, so both are isomorphic to the affine hyperbola over $\mathbf{C}^* \subset \mathbf{C}$ (after change of coordinates to $x+iy, x-iy$ on $\mathbf{C}^2$). Note that $\mathrm{O}(2, \mathbf{R})$ are the two unit circles in the $\mathbf{C}^*$'s.

$\mathrm{Sp}(2n, \mathbf{C})$ is the symplectic group, defined as for the orthogonal group, with a non-degenerate skew-symmetric matrix S (necessarily of even rank).

The Elliptic Curve (A Projective Group Variety): Take:

$$E = V(x_2^2 x_0 - x_1(x_1 - x_0)(x_1 - \lambda x_0)) \subset \mathbf{CP}^2$$

as in §4, with $\lambda \neq 0, 1$. Then E is an Abelian group:

- The additive identity is $0 = (0:1:0) \in E$ (the unique point of $E - U_0$).
- The additive inverse is $-(a_0 : a_1 : a_2) := (a_0 : a_1 : -a_2)$.

i.e. $- : U_0 \to U_0$ is the reflection of $U_0 = V(y^2 - x(x-1)(x-\lambda)) \subset \mathbf{C}^2$ about the x-axis. Note that aside from 0, the fixed points of this involution of E are the three intersection points with the x-axis: $(1:0:0), (1:1:0), (1:\lambda:0)$.

- The addition is:

$$a + b + c = 0$$

whenever $a, b, c \in E$ are on the same projective line $L \subset \mathbf{CP}^2$.

Remark: A line is an embedding

$$\Phi : \mathbf{CP}^1 \to \mathbf{CP}^2; (s:t) \mapsto (a_0 s + b_0 t : a_1 s + b_2 t : a_2 s + a_3 t)$$

If we pull back the equation of E, we get a homogeneous polynomial $P(s,t)$ of degree 3, which has three linear factors (it isn't zero because E is irreducible) and a, b, c are by definition the images of the 3 roots (which may coincide).

Example: Consider the line:

$$(s:t) \mapsto (s:t:t)$$

Then $P(s,t) = st^2 - t(t-s)(t-\lambda s)$ has roots $(1:0), (\omega_1 : 1)$ and $(\omega_2 : 1)$ where ω_1, ω_2 are the roots of $\lambda s^2 - (2+\lambda)s + 1$, and:

$$(1:0:0) + (\omega_1 : 1 : 1) = (\omega_2 : 1 : -1)$$

We will revisit the elliptic curve, and later prove some:

Non-Obvious Claims: (1) Addition defined in this way is a regular map.

(2) Addition is associative (it is obviously commutative).

We will evetually also see this, and also that, as topological groups,

$$E \cong S^1 \times S^1$$

which explains why there are 4 points of order 2.

The Grassmannian: For $m < n$, consider the rational map:

$$\Phi : \mathbf{CP}^{mn-1} \dashrightarrow \mathbf{CP}^{\binom{n}{m}-1}$$

given by "determinants." That is, if the coordinates of $\mathbf{CP}^{mn-1}$ are:

$$\{x_{ij} \mid i \in \{1, ..., m\} \text{ and } j \in \{1, ..., n\}\}$$

which we regard as the entries of an $m \times n$ matrix, then Φ is given by:

$$\Phi = (\cdots : \det(x_{ij})_{j \in J} : \cdots)$$

indexed by multi-indices $J = (j_1 < j_2 < ... < j_m)$, again arranged in some order (usually lexicographic) and x_J are the corresponding coordinates on $\mathbf{CP}^{\binom{n}{m}-1}$. By Proposition 4.5, we see that Φ is defined exactly on the locus:

$$\mathbf{CP}^{mn-1} - \bigcap_J V(\det(x_{ij})_{j \in J})$$

which is to say that Φ is defined on the locus of injective $m \times n$ matrices.

The image of Φ in $\mathbf{CP}^{\binom{n}{m}-1}$ is the *Grassmann variety* $G(m, n)$. Some linear algebra shows that $\Phi(a_{ij}) = \Phi(b_{ij})$ if and only if $\alpha A = B$ for some $\alpha \in \mathrm{GL}(m, \mathbf{C})$. Thus the points of $G(m, n)$ are in a bijection with the subspaces of $\mathbf{C}^n$ of dimension m. (In other words, the Grassmann variety is the higher-rank analogue of $\mathbf{CP}^{n-1} = \{$ lines in $\mathbf{C}^n\}$).

For each $J = (j_1 < j_2 < ... < j_m)$, consider the locally closed subset:

$$V_J := \{(... : a_{ij} : ..) | (a_{ij})_{j \in J} = I_m \text{ (the identity matrix)}\} \subset \mathbf{CP}^{mn-1}$$

This is isomorphic, on the one hand, to $\mathbf{C}^{m(n-m)}$ (the remaining coordinates) and on the other hand, to its image $\Phi(V_J) \subset \mathbf{CP}^{\binom{n}{m}-1}$, which is *closed* as a subset of $\mathbf{CP}^{\binom{n}{m}-1} - V(x_J)$. It follows, therefore, that $G(m, n)$ is a projective variety (i.e. it is closed in $\mathbf{CP}^{\binom{n}{m}-1}$) since it is covered by "basic open sets" $G(m, n) - V(x_J)$ that are closed in $\mathbf{CP}^{\binom{n}{m}-1} - V(x_J)$ (a set is closed if its intersection with each element of an open cover is closed!). And like $\mathbf{CP}^{n-1}$, the basic open sets are each isomorphic to $\mathbf{C}^{m(n-m)}$.

Example: $m = 2, n = 4$. When $J = (1,2)$, we get:

$$\mathbf{C}^4 \cong V_J = \{(1:0:1:0:a_{13}:a_{23}:a_{14}:a_{24})\} \subset \mathbf{CP}^7$$

and (for the lexicographic ordering $((1,2),(1,3),(1,4),(2,3)...)$ on the J's:

$$\Phi(V_J) = \{(1:a_{23}:a_{24}:-a_{13}:-a_{14}:a_{13}a_{24}-a_{14}a_{23})\} \subset \mathbf{CP}^5$$

and it follows that:

$$G(2,4) = V(x_{(1,2)}x_{(3,4)} - x_{(1,3)}x_{(2,4)} + x_{(1,4)}x_{(2,3)}) \subset \mathbf{CP}^5$$

is the non-degenerate quadric hypersurface in $\mathbf{CP}^5$.

In general, the ideal $I(G(m,n))$ is generated by the "Plücker" quadrics. To see them, define linear forms y_I for each m-tuple $I = (i_1, ..., i_m)$ of integers $1 \le i_k \le m$ by:

$$y_I := \begin{cases} 0 & \text{if some } i_j = i_k \text{ for } j \ne k, \text{ otherwise} \\ \operatorname{sgn}(\sigma)x_J & \text{for } \sigma(I) = J, \text{ a multi-index} \end{cases}$$

where $\operatorname{sgn}(\sigma)$ is the sign (± 1) of the unique permutation σ of $\{1, ..., m\}$ taking I to J. For each pair of m-tuples I, K and each $s = 1, ..., m$, define:

$$Q_{IKs} := y_I y_K - \sum_{t=1}^{m} y_{(i_1,\dots,i_{s-1},k_t,i_{s+1},\dots,i_m)} y_{(k_1,\dots,k_{t-1},i_s,k_{t+1},\dots,k_m)}$$

and then it turns out that $I(G(m,n)) = \langle Q_{IKs} \rangle$. Notice that already in the $m = 2, n = 4$ case this produces a lot of quadrics, but they are all the same (up to a sign)!

Examples: (a) (Projective space) When $m = 1$, there are no non-trivial Plücker quadrics, and indeed this is a good reality check, as:

$$G(1,n) = \mathbf{CP}^{n-1}$$

does in fact agree with our convention that $\mathbf{CP}^{n-1}$ is the set of lines in $\mathbf{C}^n$.

(b) (The "dual" projective space) When $m = n - 1$, there are also no non-trivial Plücker quadrics, so once again $G(n-1,n) = \mathbf{CP}^{n-1}$, though in this case the map to the Grassmannian:

$$\mathbf{CP}^{n^2-n-1} - \bigcap V((x_{ij})_{j \in J}) \to G(n-1,n) = \mathbf{CP}^{n-1}$$

is more complicated.

It is not surprising that this should be so, since a hyperplane in $\mathbf{C}^n$ is a line in the dual vector space $(\mathbf{C}^n)^v$, so this $\mathbf{CP}^{n-1}$ is the projective space of lines in $(\mathbf{C}^n)^v$, and an isomorphism from this Grassmannian to the other depends upon a choice of isomorphism $\mathbf{C}^n \cong (\mathbf{C}^n)^v$.

Notice that we now have interpetations of non-degenerate quadrics in $\mathbf{CP}^2$ (the conic, isomorphic to $\mathbf{CP}^1$, or the simplest Veronese variety), in $\mathbf{CP}^3$ (the Segre embedding $\mathbf{CP}^1 \times \mathbf{CP}^1$) and in $\mathbf{CP}^5$ (the Grassmann $G(2,4)$).

The Grassmannian is the model for moduli spaces in algebraic geometry. It parametrizes the subspaces of a vector space in the stong sense that families of such subspaces over a base variety B are the "same thing" as regular maps $\Phi : B \to G(m,n)$. This universal property of the Grassmannian generalizes in fascinating and subtle ways to other parametrization problems, such as the problem of parametrizing all the projective subvarieties $X \subset \mathbf{CP}^n$ of a given projective space with a given Hilbert polynomial, or the problem of parametrizing all the line bundles (or vector bundles) on a given complex projective variety, or quite simply the problem of parametrizing all (smooth) projective varieties themselves. This is one of the most beautiful and active areas of research in algebraic geometry, whose modern study was pioneered in the work of Grothendieck, Mumford and others.

Determinantal Varieties: For each $0 < r < m, n$ the algebraic sets:

$$D_r(m,n) := V(\langle \det(x_{ij})_{i \in I, j \in J} \rangle) \subset \mathbf{CP}^{mn-1}$$

(as $I = (i_1 < ... < i_{r+1})$ and $J = (j_1 < ... < j_{r+1})$ range over all multi-indices) are the *determinantal varieties*, with equations given by the determinants of all the $r+1 \times r+1$ minors of (x_{ij}). We will see later that these are really irreducible, but notice for now that, as a set:

$$D_r(m,n) = \{m \times n \text{ matrices of rank } \le r\} \subset \mathbf{CP}^{mn-1}$$

For example, we've already seen that the determinantal variety $D_{m-1}(m,n)$ (for $m < n$) is the locus where the rational map $\Phi : \mathbf{CP}^{mn-1} --> G(m,n)$ to the Grassmann variety is undefined. Also, we know that the determinantal variety $D_1(m,n)$ is the Segre embedding $S_{m-1,n-1} = \mathbf{CP}^{m-1} \times \mathbf{CP}^{n-1}$ since the equations $x_{ij}x_{kl} - x_{ik}x_{jl}$ for the Segre embedding are exactly the determinants of the 2×2 minors of the matrix (x_{ij})!

Special Determinantal Varieties: Determinantal varieties for symmetric and skew-symmetric matrices are defined similarly. In the symmetric case,

$$D_r(n^2) = \{n \times n \text{ symmetric matrices of rank} \leq r\} \subset \mathbf{CP}^{\binom{n+1}{2}-1}$$

where $\mathbf{CP}^{\binom{n+1}{2}-1}$ has coordinates x_{ij} for $i \leq j$. In the skew-symmetric case,

$$D_r(n \wedge n) = \{n \times n \text{ skew-symmetric matrices of rank} \leq r\} \subset \mathbf{CP}^{\binom{n}{2}-1}$$

where $\mathbf{CP}^{\binom{n}{2}-1}$ has coordinates x_{ij} for $i < j$.

Notice that $D_{2m+1}(n \wedge n) = D_{2m}(n \wedge n)$ since every skew-symmetric matrix has even rank. When n is even, the determinant of (x_{ij}) is a square(!) so although $D_{n-2}(n \wedge n) = V(\det(x_{ij}))$, we have $I(D_{n-2}(n \wedge n)) = \langle \mathrm{Pf}(x_{ij}) \rangle$ where $\mathrm{Pf}(x_{ij})$ is the Pfaffian square root of the $\det(x_{ij})$, which is irreducible.

Examples: (a) Symmetric 3×3 matrices can be given coordinates:

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{bmatrix}$$

and then the determinants of the 2×2 minors give:

$$D_1(3^2) = V(x_0x_3 - x_1^2, x_0x_4 - x_1x_2, x_0x_5 - x_2^2, x_1x_4 - x_2x_3, x_1x_5 - x_2x_4, x_3x_5 - x_4^2)$$

which are the same as the equations for the Veronese surface $V_{2,2} \subset \mathbf{CP}^5$! More generally, we always have $D_1(n^2) = V_{n-1,2} \subset \mathbf{CP}^{\binom{n+1}{2}-1}$.

(b) Skew-symmetric 4×4 matrices can be given coordinates:

$$\begin{bmatrix} 0 & x_{(1,2)} & x_{(1,3)} & x_{(1,4)} \\ -x_{(1,2)} & 0 & x_{(2,3)} & x_{(2,4)} \\ -x_{(1,3)} & -x_{(2,3)} & 0 & x_{(3,4)} \\ -x_{(1,4)} & -x_{(2,4)} & -x_{(3,4)} & 0 \end{bmatrix}$$

and then the determinant is $(x_{(1,2)}x_{(3,4)} - x_{(1,3)}x_{(2,4)} + x_{(1,4)}x_{(2,3)})^2$ so:

$$D_2(4 \wedge 4) = V(x_{(1,2)}x_{(3,4)} - x_{(1,3)}x_{(2,4)} + x_{(1,4)}x_{(2,3)})$$

is the same equation as for the Grassmannian $G(2,4) \subset \mathbf{CP}^5$. More generally, we always have $D_2(n \wedge n) = G(2,n) \subset \mathbf{CP}^{\binom{n}{2}-1}$.

Exercises 5.

1. If $X \subset \mathbf{CP}^n$, let $X = X_{n,d} \subset \mathbf{CP}^{\binom{n}{d}-1}$ be the re-embedding of X as its image under the d-uple embedding v_d. Prove that:

$$\mathbf{C}[X_{n,d}] = \mathbf{C}[X]^{(d)} = \bigoplus_{k=0}^{\infty} \mathbf{C}[X]_{kd}$$

as graded rings, and conclude that the constant term and the degree of the Hilbert polynomials of $\mathbf{C}[X]$ and of $\mathbf{C}[X_{n,d}]$ are the same, but that the leading coefficient, for instance, is multiplied by d^n.

Homogeneous coordinate rings and their Hilbert polynomials!

Representations

Rank 1 matrices, and rank 2 skew matrices

Secant lines