Math 6130 Notes. Fall 2002.

8. Dimension. We wish to define the dimension of a variety Y, and we want our definition to agree with the usual (complex) dimension of Y when Y is a complex manifold. In this case, the strange properties of the Zariski topology work in our favor, giving a simple topological definition of the "Noetherian" codimension of an irreducible closed subset $Z \subset Y$, and the Noetherian dimension of Y is then the codimension of a point. In this section, we will explore some applications of dimension, and prove the important:

Dimension Theorem 8.1: If Y is a quasi-projective variety, then:

(a) The Noetherian dimension of Y and

(b) The transcendence degree of the field $\mathbf{C}(Y)$ over $\mathbf{C}$

are the same. And if $Y \subseteq \mathbf{CP}^n$ is projective, then they are also the same as:

(c) The degree of the Hilbert polynomial of $\mathbf{C}[Y]$.

Definition: The *Noetherian codimension* of an irreducible closed set $Z \subset Y$ is the maximal length c of all (proper) chains of irreducible closed sets:

$$Z = Z_0 \subset Z_1 \subset Z_2 \subset ... \subset Z_c = Y$$

We will write $\text{cod}_Y(Z)$ for the codimension of Z in Y.

Remarks: (a) If $U \subset Y$ is an open subset and $U \cap Z \neq \emptyset$, then:

$$\text{cod}_Y(Z) = \text{cod}_U(U \cap Z)$$

since taking closures is a bijection between irreducible closed subsets of U and irreducible closed subsets of Y that intersect U. As a consequence of this, we can always compute codimension on an affine open subset of Y.

(b) If Y is affine and $Z \subset Y$ has codimension 1, then there is a regular function $\overline{f} \in \mathbf{C}[Y]$ so that Z is one of the irreducible components:

$$Z_1 \cup ... \cup Z_n = V(\overline{f}) \subset Y$$

Indeed, let $\overline{f}$ be any (nonzero) function in $I(Z)$. Then $Z \subseteq V(\overline{f})$, and because Z is irreducible, it must be contained in one of the Z_i. But because it has codimension 1, it must be *equal* to one of the Z_i.

We'd like to have the converse to (b), telling us that every component of *every* hypersurface $V(\overline{f}) \subset Y$ has codimension 1. For this we will need another reminder from field theory and some commutative algebra;

Field Theory II: If $K \subset L$ is a finite extension of fields and $\alpha \in L$, then

$$\mathrm{Nm}_{L/K}(\alpha) = \det(m_\alpha) \in K \text{ and } \mathrm{Tr}_{L/K}(\alpha) = \mathrm{tr}(m_\alpha) \in K$$

are the determinant and trace of the linear transformation $m_\alpha : L \to L$ given by multiplication by α. They are (up to a sign) the constant and next-to-highest coefficients of the *characteristic polynomial* $\det(\lambda I - m_\alpha)$, which is a polynomial of degree $[L : K]$, the dimension of L as a K-vector space, and is always a power of the *minimal polynomial* of α in $K[\lambda]$. Note in particular that $\mathrm{Nm}_{L/K}(\alpha\beta) = \mathrm{Nm}_{L/K}(\alpha)\mathrm{Nm}_{L/K}(\beta)$, and $\mathrm{Nm}_{L/K}(\alpha) = \alpha^{[L:K]}$ if $\alpha \in K$.

If $K \subset L$ are the fields of fractions of Noetherian domains $A \subset B$, such that A is a UFD and B is finitely generated as an A-module, then it follows from Gauss' lemma that the minimal polynomial of each $\alpha \in B$ is in $A[\lambda]$. In particular, $\mathrm{Nm}_{L/K}(\alpha)$ and $\mathrm{Tr}_{L/K}(\alpha)$ are in A whenever $\alpha \in B$.

Krull's Principal Ideal Theorem:

If $\mathrm{trd}_{\mathbf{C}}(\mathbf{C}(Y)) = d$ for an affine variety Y, and if $\overline{f} \in \mathbf{C}[Y]$ is nonzero, then $\mathrm{trd}_{\mathbf{C}}(\mathbf{C}(Z_i)) = d - 1$ for each of the irreducible components $Z_i \subseteq V(\overline{f})$.

Proof: In the simple case $Y = \mathbf{C}^d$, then each $Z_i = V(f_i)$ for a prime polynomial f_i in the factorization of $f \in \mathbf{C}[x_1, ..., x_d]$, and then $\mathbf{C}(Z_i)$ is the field of fractions of $\mathbf{C}[x_1, ..., x_d]/f_i$, and the theorem is just the second property of trd from §1. We prove the theorem by reducing it to this case.

First, recall that $\mathbf{C}(U) = \mathbf{C}(Y)$ for an open subsets $U \subset Y$, so given Z_i, we can replace Y with a basic open set $Y - V(\overline{g})$ that intersects Z_i but does not intersect any of the other Z_j. Thus we may assume that $Z = Z_i = V(\overline{f})$. The advantage is that now the prime ideal $I(Z) = \sqrt{\langle \overline{f} \rangle}$ by the Nullstellensatz.

Next let $\Phi : Y \to \mathbf{C}^d$ be the finite dominant map from affine Noether Normalization (§7) with $\Phi^* : \mathbf{C}[y_1, ..., y_d] \subset \mathbf{C}[Y]$. The image $\Phi(Z) \subseteq \mathbf{C}^n$ is closed (and irreducible) by Proposition 7.5, and the map $\Phi|_Z : Z \to \Phi(Z)$ is also finite and dominant, so the field extension $\mathbf{C}(\Phi(Z)) \subset \mathbf{C}(Z)$ is finite, and $\mathrm{trd}_{\mathbf{C}}(\mathbf{C}(Z)) = \mathrm{trd}_{\mathbf{C}}(\mathbf{C}(\Phi(Z)))$. So it suffices to prove, using the simple case above, that $\Phi(Z) \subset \mathbf{C}^d$ is hypersurface.

Let $K = \mathbf{C}(y_1, ..., y_d), L = \mathbf{C}(Y)$, and let $g = \text{Nm}_{L/K}(\overline{f}) \in \mathbf{C}[y_1, ..., y_d]$. If $\lambda^m + g_{m-1}\lambda^{m-1} + ... + g_0$ is the characteristic polynomial of $\overline{f} \in \mathbf{C}[Y]$ then all coefficients are in $\mathbf{C}[y_1, ..., y_d]$ and $g = \pm g_0$, so:

$$g = \mp\overline{f}(\overline{f}^{m-1} + g_{m-1}\overline{f}^{m-2} + ... + g_1) \in I(Z) \subset \mathbf{C}[Y]$$

so $g \in I(Z) \cap \mathbf{C}[y_1, ..., y_d]$ and then $\sqrt{\langle g \rangle} \subseteq I(Z) \cap \mathbf{C}[y_1, ..., y_d] \subset \mathbf{C}[y_1, ..., y_d]$ since $I(Z) \cap \mathbf{C}[y_1, ..., y_d]$ is prime. We know $\Phi(Z) = V(I(Z) \cap \mathbf{C}[y_1, ..., y_d])$ (see the proof of Proposition 7.5(b)) so we are done if we can show that $\sqrt{\langle g \rangle} = I(Z) \cap \mathbf{C}[y_1, ..., y_d]$ because then $\Phi(Z) = V(g) \subset \mathbf{C}^d$ is a hypersurface. But if $h \in I(Z) \cap \mathbf{C}[y_1, ..., y_d] = \sqrt{\langle \overline{f} \rangle} \cap \mathbf{C}[y_1, ..., y_d]$, then some $h^M = \overline{k} \cdot \overline{f}$ for $\overline{k} \in \mathbf{C}[Y]$, so $h^{M[L:K]} = \text{Nm}_{L/K} h^M = (\text{Nm}_{L/K}\overline{k})g$, and then $h \in \sqrt{\langle g \rangle}$. Thus $I(Z) \cap \mathbf{C}[y_1, ..., y_d] \subseteq \sqrt{\langle g \rangle}$ so we really are done.

Proposition 8.2: (a) Every irreducible component $Z \subset V(\overline{f}) \subset Y$ of every hypersurface in Y has codimension 1.

(b) A chain $Z = Z_0 \subset Z_1 \subset ... \subset Z_c = Y$ of closed irreducible subsets is maximal if and only if each $Z_i \subset Z_{i+1}$ has Noetherian codimension 1.

(c) The codimension of a point $p \in Y$ is equal to $\text{trd}_{\mathbf{C}}(\mathbf{C}(Y))$ and is, in particular, independent of the choice of the point p.

(d) If $Z \subset Y \subset X$ are closed and irreducible, then

$$\text{cod}_X(Y) = a \text{ and } \text{cod}_Y(Z) = b \Rightarrow \text{cod}_X(Z) = a + b$$

Proof: (a) By Krull's theorem, $\text{trd}_{\mathbf{C}}(\mathbf{C}(Z)) = \text{trd}_{\mathbf{C}}(\mathbf{C}(Y)) - 1$. If $\text{cod}_Y(Z) > 1$, there would be a chain $Z \subset Z' \subset Y$ with $\text{cod}_Y(Z') = 1$. By Remark (b), $Z' \subset V(\overline{f}')$ for some $\overline{f}'$, so $\text{trd}_{\mathbf{C}}(\mathbf{C}(Z')) = \text{trd}_{\mathbf{C}}(\mathbf{C}(Y)) - 1$ (again by Krull), and then $\text{trd}_{\mathbf{C}}(\mathbf{C}(Z)) < \text{trd}_{\mathbf{C}}(\mathbf{C}(Y)) - 1$, a contradiction.

(b) If $Z = Z_0 \subset Z_1 \subset ... \subset Z_c = Y$ is any maximal chain, then each $Z_i \subset Z_{i+1}$ has codimension 1, otherwise we could construct a longer chain! Conversely, if each $Z_i \subset Z_{i+1}$ has codimension 1, then by Remark (b) and Krull's theorem, $\text{trd}_{\mathbf{C}}(\mathbf{C}(Z_i)) = \text{trd}_{\mathbf{C}}(\mathbf{C}(Z_{i+1})) - 1$, so $\text{trd}_{\mathbf{C}}(Z) = \text{trd}_{\mathbf{C}}(Y) - c$ for any such chain, and so each one is maximal, of length $\text{trd}_{\mathbf{C}}(Y) - \text{trd}_{\mathbf{C}}(X)$.

(c) This is immediate from the last sentence in the proof of (b).

(d) Fill $Z \subset Y$ with a maximal chain of length a and $Y \subset X$ with a maximal chain of length b. It follows from (b) that the concatentation of the two chains is maximal, of length $a + b$. Alternatively, this is also immediate from the last sentence of the proof of (b).

Definition: The *dimension* of Y is the Noetherian codimension of a point.

Note: By Remark (a), the Proposition holds for quasi-projective varieties. Thus we have proved (a) = (b) in the Dimension Theorem, and then we can (and frequently do) use $\dim(Y) = \mathrm{trd}_{\mathbf{C}}(\mathbf{C}(Y))$ to compute dimension. It now follows immediately from Proposition 8.2(d) that, as expected:

$$\mathrm{cod}_Y(Z) = \dim(Y) - \dim(Z)$$

Examples: (a) The dimension of $\mathbf{C}^n$ (and therefore of $\mathbf{CP}^n$) is n.

(b) If $\dim(X) = m$ and $\dim(Y) = n$, then $\dim(X \times Y) = m + n$.

To see this, we may assume that X and Y are affine, and then use the result $\mathbf{C}[X \times Y] = \mathbf{C}[X] \otimes_{\mathbf{C}} \mathbf{C}[Y]$ of Exercise 6.1. By Noether normalization, $\mathbf{C}[x_1, ..., x_m] \subset \mathbf{C}[X]$ and $\mathbf{C}[y_1, ..., y_n] \subset \mathbf{C}[Y]$ are finitely generated modules, so $\mathbf{C}[X] \otimes \mathbf{C}[Y]$ is a finitely generated module over $\mathbf{C}[x_1, ..., x_m, y_1, ..., y_n]$ (generated by the tensors of the two sets of generators). Hence $\mathbf{C}(X \times Y)$ is a finite field extension of $\mathbf{C}(x_1, ..., x_m, y_1, ..., y_n)$ and so $\dim(X \times Y) = m + n$.

(c) If $X \subset \mathbf{CP}^n$ is a projective variety then $\dim(C(X)) = \dim(X) + 1$ where $C(X) \subset \mathbf{C}^{n+1}$ is the affine cone over X (from Exercise 6.2). This follows from Exercise 6.2 and Example (b) above.

(d) If $\Phi : X \to Y$ is dominant, then $\dim(X) - \dim(Y) = \mathrm{trd}_{\mathbf{C}(Y)}(\mathbf{C}(X))$ for the inclusion $\Phi^* : \mathbf{C}(Y) \hookrightarrow \mathbf{C}(X)$. In particular, $\dim(X) \geq \dim(Y)$.

Proposition 8.3: If $X \subset \mathbf{C}^n$ and $Y \subset \mathbf{C}^n$ are closed subvarieties with $\mathrm{cod}_{\mathbf{C}^n}(X) = a$ and $\mathrm{cod}_{\mathbf{C}^n}(Y) = b$, then $\mathrm{cod}_{\mathbf{C}^n}(Z) \leq a+b$ for every component $Z \subset X \cap Y$ (if $X \cap Y$ is empty, this is vacuously true!)

Proof: Consider first the simple case $X = V(\langle f_1, ..., f_a \rangle)$, i.e. the case where X has the "right number of equations." Then we get a chain:

$$Z = Z_a \subset Z_{r-1} \subset ... \subset Z_1 \subset Z_0 = Y$$

where each Z_i is an irreducible component of $Z_{i-1} \cap V(\overline{f}_i)$. Now either $Z_i = Z_{i-1}$, or else $Z_i \subset Z_{i-1}$ has codimension 1 by Krull's theorem, so by Proposition 8.2(b) $\mathrm{cod}_Y(Z) \leq a$, and by Proposition 8.2(d) $\mathrm{cod}_{\mathbf{C}^n}(Z) \leq a+b$.

Now for the general case. Let $\Delta \subset \mathbf{C}^{2n}$ be the diagonal. Then $\Delta \cong \mathbf{C}^n$ (by either of the two projections) and under either projection:

$$\pi : (X \times Y) \cap \Delta \to X \cap Y$$

is a bijection. It follows that the components of $(X \times Y) \cap \Delta$ are carried isomorphically to the components of $X \cap Y$. So it suffices to prove that $\dim(Z) \geq n - (a+b)$ for every component $Z \subset (X \times Y) \cap \Delta$.

But $\Delta \subset \mathbf{C}^{2n}$ has the right number of equations! If $\mathbf{C}[x_1, ..., x_n, y_1, ..., y_n]$ is the coordinate ring of $\mathbf{C}^{2n}$, then $\Delta = V(\langle x_1 - y_1, ..., x_n - y_n \rangle)$. So the simple case above applies to components Z of $X \times Y \cap \Delta$, which then satisfy:

$$\mathrm{cod}_{\mathbf{C}^{2n}}(Z) \leq \mathrm{cod}_{\mathbf{C}^{2n}}(X \times Y) + \ \mathrm{cod}_{\mathbf{C}^{2n}}(\Delta) = (a+b) + n$$

and so each such Z has the desired $\dim(Z) \geq 2n - ((a+b)+n) = n - (a+b)$.

Remark: It is natural to ask whether Proposition 8.3 holds when $\mathbf{C}^n$ is replaced by another variety. It does hold when $\mathbf{C}^n$ is replaced by $\mathbf{CP}^n$, or any quasi-projective variety that is covered by open subsets that are isomorphic to open subsets of $\mathbf{C}^n$. But it doesn't hold in general, even for affine varieties! For example, consider the affine variety $W = V(x_0x_3 - x_1x_2) \subset \mathbf{C}^4$. The planes $V(x_0) \cap V(x_1)$ and $V(x_2) \cap V(x_3) \subset W$ both have codimension 1 in W but their intersection is the origin, which has codimension 3.

We will think more about this question in §9.

Corollary 8.4: If two projective varieties $X, Y \subset \mathbf{CP}^n$ ought to intersect, in the sense that $\mathrm{cod}_{\mathbf{CP}^n}(X) + \mathrm{cod}_{\mathbf{CP}^n}(Y) \leq n$, then they do intersect.

Proof: Consider the affine cones $C(X), C(Y) \subset \mathbf{C}^{n+1}$. Example (c) gives $\mathrm{cod}_{\mathbf{CP}^n}(X) = \mathrm{cod}_{\mathbf{C}^{n+1}}(C(X))$ and $\mathrm{cod}_{\mathbf{CP}^n}(Y) = \mathrm{cod}_{\mathbf{C}^{n+1}}(C(Y))$, so if $\mathrm{cod}_{\mathbf{CP}^n}(X) + \mathrm{cod}_{\mathbf{CP}^n}(Y) \leq n$ then $\mathrm{cod}_{\mathbf{C}^{n+1}}(C(X)) + \mathrm{cod}_{\mathbf{C}^{n+1}}(C(Y)) \leq n$.

The origin $0 \in C(X) \cap C(Y)$ is always in the intersection of two cones, and if there is any other point in the intersection, then each component of $C(X) \cap C(Y)$ is the affine cone over a component of $X \cap Y \subset \mathbf{CP}^n$. Thus we need to show that $C(X) \cap C(Y)$ contains a point other than 0. But by Proposition 8.3, each component $Z \subset C(X) \cap C(Y)$ has $\mathrm{cod}_{\mathbf{C}^{n+1}}(Z) \leq n$, so $\dim(Z) \geq 1$ and Z contains points other than 0.

Next, we use dimension to help us analyze dominant regular maps.

Proposition 8.5: A dominant map $\Phi : X \to Y$ of quasi-projective varieties is *birational* if and only if there is an open subset $U \subset Y$ such that the restriction $\Phi : \Phi^{-1}(U) \to U$ is an isomorphism.

Proof: Recall that birational means dominant and $\Phi^* : \mathbf{C}(Y) = \mathbf{C}(X)$. If there is such a U, this is clearly the case. Given a birational Φ, to find the U we can assume that Y is affine, replacing Y with any affine open subset.

First, we will find $U_0 \subset Y$ and $V_0 \subset \Phi^{-1}(U_0)$ such that $\Phi : V_0 \to U_0$ is an isomorphism. For this, we can assume that X is affine. Then the inclusion $\Phi^* : \mathbf{C}[Y] \hookrightarrow \mathbf{C}[X]$ may not be itself an equality, but if we write $\mathbf{C}[X] = \mathbf{C}[x_1, ..., x_n]/P$, then each $x_i = \frac{\overline{f}_i}{\overline{g}_i} \in \mathbf{C}(Y)$ and if $\overline{g} = \prod \overline{g}_i \in \mathbf{C}[Y]$, then $\mathbf{C}[Y]_{\overline{f}} = \mathbf{C}[X]_{\overline{f}}$. So take $U_0 = Y - V(\overline{f})$ and $V_0 = \Phi^{-1}(U_0) = X - V(\overline{f})$.

For the general case, start with the open sets $U_0 \subset Y$ and $V_0 \subset X$. Each irreducible component $Z_i \subset \Phi^{-1}(U_0) - V_0$ has $\dim(Z_i) < \dim(V_0)$ so $\dim(\overline{\Phi(Z)}) \leq \dim(Z) < \dim(U_0)$. In particular, $\cup\overline{\Phi(Z_i)} \subset U_0$ is *not* equal to U_0, so we may take $U = U_0 - \cup\overline{\Phi(Z_i)}$ and then $\Phi^{-1}(U) = V_0 - \cup\Phi^{-1}(\overline{\Phi(Z_i)})$ and then $\Phi : \Phi^{-1}(U) \to U$ is an isomorphism, as desired.

Example: Consider the regular map:

$$\Phi : S_{1,1} - \{(0:0:0:1)\} \to \mathbf{CP}^2; \quad (a:b:c:d) \mapsto (a:b:c)$$

where $S_{1,1} = V(x_0x_3 - x_1x_2) \subset \mathbf{CP}^3$. Then the image of Φ is the set:

$$\{(1:b:c)\} \cup \{(0:1:0)\} \cup \{(0:0:1)\} \subset \mathbf{CP}^2$$

and the first set is open (it will be our U) and isomorphic to $\mathbf{C}^2$ and:

$$\Phi^{-1}(U) = \Phi^{-1}\{(1:b:c)\} = \{(1:b:c:bc)\} \subset S_{1,1}$$

is $\mathbf{C}^1 \times \mathbf{C}^1 \subset \mathbf{CP}^1 \times \mathbf{CP}^1$ of Example (a) after Corollary 6.3. In particular, in this case $\Phi : \Phi^{-1}(U) \to U$ is an isomorphism from $\mathbf{C}^1 \times \mathbf{C}^1 \subset \mathbf{CP}^1 \times \mathbf{CP}^1$ to $\mathbf{C}^2 \subset \mathbf{CP}^2$ although $\mathbf{CP}^1 \times \mathbf{CP}^1$ and $\mathbf{CP}^2$ are not themselves isomorphic.

So birational maps are "almost" isomorphisms. For arbitrary dominant maps $\Phi : X \to Y$, the idea is that if $r = \dim(X) - \dim(Y) = \mathrm{trd}_{\mathbf{C}(Y)}(\mathbf{C}(X))$, then the fibers of Φ ought to be a union of r-dimensional varieties. We'll see that this holds over an open set (analogous to the set U of Proposition 8.5), and that in general, the fiber dimensions are at least r. More precisely:

Definition: A real-valued function $e : X \to \mathbf{R}$ from a topological space X is *upper-semicontinuous* if for each $\alpha \in \mathbf{R}$, the subset:

$$U_\alpha := \{x \in X \mid e(x) < \alpha\} \subset X$$

is open (i.e. the value of e jumps up only on closed sets).

Theorem 8.6: If $\Phi : X \to Y$ is dominant and $r = \mathrm{trd}_{\mathbf{C}(Y)}\mathbf{C}(X)$, then:

(a) If $p \in Y$ then the dimension of every component $Z \subseteq \Phi^{-1}(p)$ of every fiber satisfies $\dim(Z) \geq r$ $(\Leftrightarrow\ \mathrm{cod}_X(Z) \leq\ \dim(Y))$.

(b) More generally, among the components $Z \subseteq \Phi^{-1}(W)$ of the preimage of a closed irreducible subset $W \subseteq Y$, every component that *dominates* W, in the sense that $\overline{\Phi(Z)} = W$, must satisfy $\mathrm{cod}_X(Z) \leq \mathrm{cod}_Y(W)$.

(c) The image $\Phi(X)$ contains an open dense subset $U \subseteq \Phi(X) \subseteq Y$ with the property that if W in (b) intersects U, then $\mathrm{cod}_X(Z) = \mathrm{cod}_Y(W)$ in (b). In particular, if $p \in U$ in (a), then $\dim(Z) = r$.

(d) The "maximum fiber dimension" function $e : X \to \mathbf{R}$ defined by:

$$e(x) = \max\{\dim(Z) \mid Z \text{ is a component of } \Phi^{-1}(\Phi(x)) \text{ and } x \in Z\}$$

is upper-semicontinuous.

Example: Consider the blow-up $\Phi : \mathbf{C}^3 \to \mathbf{C}^3; \Phi(a, b, c) = (a, ab, ac)$. The image of Φ is $(\mathbf{C}^3 - \{(0, t, u)\}) \cup 0$ and the fibers of Φ are:

$\Phi^{-1}(0) = \{(0, b, c)\}$ (dimension 2)

$\Phi^{-1}(s, t, u) = \{(s, ts^{-1}, us^{-1})\}$ for all other points of the image.

Thus the dimension of each fiber is 0 or 2, and if $(s, t, u) \in U = \Phi(\mathbf{C}^3) - 0$, then the dimension is $0 = r$. The upper semi-continuous $e(x)$ is then:

$$e(a, b, c) = \begin{cases} 0 \text{ if } a \neq 0 \\ 2 \text{ if } a = 0 \end{cases}$$

Note that $\Phi^{-1}(\{(s, 0, 0)\}) = \{(a, 0, 0)\} \cup \{(0, b, c)\}$ has two components, but only $\{(a, 0, 0)\}$ dominates $\{(s, 0, 0)\}$, so only that component is covered by the theorem. This is good since the line $\{(s, 0, 0)\}$ intersects U and the other component has codimension 1 instead of $\mathrm{cod}_{\mathbf{C}^3}(\{(s, 0, 0)\}) = 2$. Note also that Φ is birational, and that $U = \mathbf{C}^3 - \{(0, t, u)\}$ (but not $\mathbf{C}^3 - 0$) works for Proposition 8.5.

Proof of (b): If $Y' \subseteq Y$ is open and affine and $Y' \cap W \neq \emptyset$, then each $Z \cap \Phi^{-1}(Y') \subset Z$ is open and nonempty (but maybe not affine) because Z dominates W. Given one Z, we can take an open affine $X' \subseteq \Phi^{-1}(Y') \subseteq X$ intersecting Z, and then it suffices to prove (a) for the induced dominant map $\Phi' : X' \to Y'$ of affine varieties. In other words, it suffices to consider the case where X and Y are affine.

Let $s = \operatorname{cod}_Y(W)$. Then by Remark (b) and induction (see Exercise 8.1) there are $\overline{f}_1, ..., \overline{f}_s \in \mathbf{C}[Y]$ such that $W \subseteq V(\langle \overline{f}_1, ..., \overline{f}_s \rangle)$ is a component. Then $Z \subseteq V(\langle \Phi^*(\overline{f}_1), ..., \Phi^*(\overline{f}_s) \rangle)$. I claim Z is a component of this set. If $Z \subseteq Z' \subseteq V(\langle \Phi^*(\overline{f}_1), ..., \Phi^*(\overline{f}_s) \rangle)$ and Z' is a component, then since we assumed $W = \overline{\Phi(Z)}$, we also have $W = \overline{\Phi(Z')} \subseteq V((\overline{f}_1, ..., \overline{f}_s))$ since $\overline{\Phi(Z')}$ is irreducible. Thus $Z' \subseteq \Phi^{-1}(W)$ is a component dominating W, and since it contains Z, it must be equal to Z. Finally, it is an easy application of Krull to see that each component of $V(\langle \Phi^*(\overline{f}_1), ..., \Phi^*(\overline{f}_s) \rangle)$ has codimension $\leq s$, and we are done. Notice that (a) is the special case of (b) where $W = p$.

Proof of (c): As in (b), we may assume Y is affine. We can also assume X is affine, with a little extra care. We may cover $X = \cup_{i=1}^n V_i$ by affine open subsets V_i. If we let $U_i \subset \Phi(V_i)$ be open subsets of Y satisfying (c) for each V_i, then $U := \cap_{i=1}^n U_i \subset \Phi(X)$ will be an open subset of Y satisfying (c) for X. So indeed we can prove this one V_i at a time, and assume X is affine. Then I claim that there are $\overline{f} \in \mathbf{C}[Y]$ and $\overline{g}_1, ..., \overline{g}_r \in \mathbf{C}[X]$ such that:

(i) The map $\alpha^* : \mathbf{C}[Y]_{\overline{f}}[y_1, ..., y_r] \to \mathbf{C}[X]_{\overline{f}}; \quad y_i \mapsto \overline{g}_i$ is injective, and

(ii) $\mathbf{C}[X]_{\overline{f}}$ is finitely generated as a $\mathbf{C}[Y]_{\overline{f}}[y_1, ..., y_r]$-module.

First, notice that (c) follows from (i) and (ii). If we let $U = Y - V(\overline{f})$ then $\Phi^{-1}(U) = X - V(\overline{f})$ and then Φ factors when restricted to $\Phi^{-1}(U)$:

$$\begin{array}{ccccc} X & & \stackrel{\Phi}{\to} & & Y \\ \cup & & & & \cup \\ \Phi^{-1}(U) & \stackrel{\alpha}{\to} & U \times \mathbf{C}^r & \stackrel{\pi}{\to} & U \end{array}$$

where α is dominant and finite and π is the projection. Given W, then $\Phi^{-1}(W \cap U) \subset \Phi^{-1}(U)$ is closed and maps onto $(W \cap U) \times \mathbf{C}^r$ with finite fibers by Proposition 7.5, and each component $Z \subset \Phi^{-1}(W \cap U)$ maps (finitely and dominantly) to its image in $(W \cap U) \times \mathbf{C}^r$, so $\dim(Z) \leq r + \dim(W)$ which translates to $\operatorname{cod}_X(Z) \geq \operatorname{cod}_Y(W)$, and (b) gave the other inequality.

To see (i) and (ii), consider the $\mathbf{C}(Y)$-algebra $\mathbf{C}[X]_S$ for $S = \mathbf{C}[Y] - 0$. If $\mathbf{C}[X] = \mathbf{C}[x_1, ..., x_n]/P$, then $\mathbf{C}[X]_S = \mathbf{C}(Y)[x_1, ..., x_n]/Q$ for some prime ideal Q, so since $\text{trd}_{\mathbf{C}(Y)}(\mathbf{C}(X)) = r$, we can apply Noether Normalization (over $\mathbf{C}(Y)$) to find $\mathbf{C}(Y)[y_1, ..., y_r] \subset \mathbf{C}[X]_S$ so $\mathbf{C}[X]_S$ is finitely generated as a $\mathbf{C}(Y)[y_1, ...y_r]$-module. We need now to replace $\mathbf{C}[X]_S$ with some $\mathbf{C}[X]_{\overline{f}}$.

To do this, we can assume that the inclusion $\mathbf{C}(Y)[y_1, ..., y_r] \hookrightarrow \mathbf{C}[X]_S$ takes each y_i to $\overline{g}_i \in \mathbf{C}[X]$ (multiplying by the denominator if needed). This gives $\mathbf{C}[Y][y_1, ..., y_r] \hookrightarrow \mathbf{C}[X]$. But $\mathbf{C}[X]$ may not be finitely generated as a $\mathbf{C}[Y][y_1, ..., y_r]$-module (i.e. the map Φ itself doesn't usually factor). On the other hand, if $\overline{h}_1, ..., \overline{h}_m \in \mathbf{C}[X]_S$ generate it as a $\mathbf{C}(Y)[y_1, ..., y_r]$-module, we can solve $x_i = \sum_{j=1}^m p_{ij}(y)\overline{h}_j$ for each of the x_i above and $\overline{h}_j\overline{h}_k = \sum c_{ijl}(y)\overline{h}_l$, where each $p_{ij}(y), c_{jkl}(y) \in \mathbf{C}(Y)[y_1, ..., y_r]$. Let $\overline{f} \in \mathbf{C}[Y]$ be the product of *all* denominators of the $\overline{h}_j$ and of all coefficients of the $p_{ij}(y)$ and $c_{jkl}(y)$. Then each $\overline{h}_j \in \mathbf{C}[X]_{\overline{f}}$ and each $p_{ij}(y), c_{jkl}(y) \in \mathbf{C}[Y]_{\overline{f}}$, and since $\mathbf{C}[X]_{\overline{f}}$ is generated by the x_i as an algebra over $\mathbf{C}[Y]_{\overline{f}}$, it follows that $\mathbf{C}[X]_{\overline{f}}$ is generated by the $\overline{h}_j$ as a $\mathbf{C}[Y]_{\overline{f}}[y_1, ..., y_r]$-module, as desired.

Proof of (d): Since $e(x)$ takes integer values, we only need to check that the sets $U_a \subset X$ are open when $a > 0$ is an integer, or equivalently that the sets $X_a := \{x \in X \mid e(x) \geq a\}$ are closed. By (a), $X_a = X$ for $a \leq r$.

Let $U \subset Y$ be any (nonempty!) open subset of $\overline{\Phi(X)}$ satisfying the conditions of (c). If $x \in \Phi^{-1}(U)$, then by (c), $e(x) = r$, so if $a > r$, it follows that $X_a \subseteq X - \Phi^{-1}(U)$ for that open set U.

Let $Z_1 \cup ... \cup Z_n = X - \Phi^{-1}(U)$ be the components (all of which have smaller dimension than X). By induction on the dimension of the domain, when we consider the maps $\Phi|_{Z_i} : Z_i \to \overline{\Phi(Z_i)}$ we may assume that each of the sets $(Z_i)_a := \{x \in Z_i \mid e(x) \geq a\} \subseteq Z_i$ is closed. If $x \in X$ and $e(x) \geq a$, there is a component of $\Phi^{-1}(\Phi(x))$ of dimension at least a passing through x and this component is contained in $X - \Phi^{-1}(U)$, so it must lie entirely in some Z_i by irreducibility. Thus we can express $X_a = \cup_{i=1}^n (Z_i)_a$ as a finite union of closed sets, so it is closed.

Remark: This theorem points out a distinctive property of regular maps, which is very far from being true for ordinary differentiable maps. In fact, Morse theory depends upon its failure! For example, consider the norm map $\Phi : \mathbf{R}^n \to \mathbf{R}; \vec{v} \mapsto ||\vec{v}||$. Then $\Phi^{-1}(\alpha) \cong S^{n-1}$ has the expected dimension when $\alpha > 0$, but $\Phi^{-1}(0) = 0$ is a point!

Finally, we turn to the Hilbert polynomial of a projective variety.

Proof of Theorem 8.1 (a),(b) ⇔ (c): Let $Y \subseteq \mathbf{CP}^n$ be a projective variety of dimension d. Then by Proposition 8.2(d), we can find a chain of projective varieties:

$$p = Z_0 \subset \ldots \subset Z_d = Y \subset Z_{d+1} \subset \ldots \subset Z_n = \mathbf{CP}^n$$

where $p \in Y$ is any point, and each $Z_i \subset Z_{i+1}$ has codimension 1. For each of these inclusions, we can find a homogeneous polynomial F_i (of some degree) such that Z_i is a component of $V(\overline{F}_i) \subset Z_{i+1}$, reasoning as in Remark (b). Then in particular, $\overline{F}_i$ is not a zero divisor in $\mathbf{C}[Z_{i+1}]$, so by Exercise 2.1(b), the Hilbert polynomials satisfy the inequality:

$$\deg(H_{\mathbf{C}[Z_i]}(d)) \le \deg(H_{\mathbf{C}[Z_{i+1}]/\langle \overline{F}_i \rangle}(d)) = \deg(H_{\mathbf{C}[Z_{i+1}]}) - 1$$

The Hilbert polynomial of $\mathbf{CP}^n$ has degree n and the Hilbert polynomial of a point has degree 0, so it follows that the Hilbert polynomial of each Z_i has degree i, which agrees with the dimension.

Looking Ahead: We will think of a projective variety $X \subseteq \mathbf{CP}^n$ as a *projective model* for its field of rational functions. We've seen so far that any two projective models X, Y of the same field admit birational maps $\Phi : X \dashrightarrow Y$ and $\Psi : Y \dashrightarrow X$ (Proposition 7.5) and then that there are open subsets $U \subset Y$ and $V \subset X$ such that $\Phi : \Phi^{-1}(U) \to U$ and $\Psi : \Psi^{-1}(V) \to V$ are isomorphisms (Proposition 8.5). So in particular, any two models of the same field can be thought of as two different compactifications of a common open set. We've also seen that the transcendence degree of a field can be read off from either the Zariski topology or the Hilbert polynomial of any projective model.

So this begs several questions. Among all the projective models of a field, is there a "best" one (or ones)? Are there other numerical invariants of a field that can be read off from geometric invariants of projective models? And what is the significance of the *coefficients* of the Hilbert polynomial? The leading coefficient is simple to understand...it is the number of intersection points of X with a "general" projective plane in $\mathbf{CP}^n$ of complementary dimension, and totally dependent upon the model. But the constant coefficient is much more mysterious and "intrinsic" to the field.

Exercises 8.

1. Use Krull's theorem to prove that the image $\Phi(\mathbf{C}^2)$ of the blow-up map $\Phi : \mathbf{C}^2 \to \mathbf{C}^2; \Phi(a,b) = (a, ab)$, with topology and sheaf induced from $\mathbf{C}^2$, is not isomorphic to any quasi-projective variety.

2. If $\Phi : X \to Y$ is dominant and $\Phi^* : \mathbf{C}(Y) \hookrightarrow \mathbf{C}(Y)$ is a finite extension, find an open subset $U \subset Y$ such that $\Phi|_{\Phi^{-1}(U)} : \Phi^{-1}(U) \to U$ is a finite map. (Hint: Look at the proof of Theorem 8.6 (c).)

3. (a) If $\Phi : \mathbf{CP}^n \to Y$ is a regular map to any quasi-projective variety, prove that either Φ is a constant map or else $\dim(\overline{\Phi(\mathbf{CP}^n)}) = n$.

(b) Prove that $\mathbf{CP}^m \times \mathbf{CP}^n$ is not isomorphic, and not even homeomorphic (in the Zariski topology) to $\mathbf{CP}^{m+n}$ when both $m, n \geq 1$.