Math 6140 Notes. Spring 2003.

11. Codimension One Phenomena. A property of the points of a variety X "holds in codimension one" if the locus of points for which the property *fails* to hold is contained in a closed subset $Z \subset X$ whose components all have codimension 2 or more. We'll see several examples of this:

• A normal variety X is non-singular in codimension 1 (Proposition 11.2).

• If X is normal then a rational map $\Phi : X - - > \mathbf{P}^n$ is regular in codimension 1 (on X) (Proposition 11.3).

• If Y is nonsingular and $\Phi : X \to Y$ is a surjective birational regular map and an isomorphism in codim 1, then Φ is an isomorphism (Prop 11.4)

Blowing up a point is a simple example of a surjective birational regular map which is not an isomorphism. We'll consider this in some detail.

Definition: If X is any variety and $Z \subset X$ is a closed subvariety, then the *stalk* of $\mathcal{O}_X$ *along* Z is the ring:

$$\mathcal{O}_{X,Z} := \bigcup_{\{U \mid U \cap Z \neq \emptyset\}} \mathcal{O}_X(U) \subseteq \mathbf{C}(X)$$

Examples: (a) $\mathcal{O}_{X,X} = \mathbf{C}(X)$

(b) If $Z = x \in X$, then $\mathcal{O}_{X,Z} = \mathcal{O}_{X,x}$ the stalk from §9.

Properties: This stalk shares many of the properties from §9.

(i) If $V \subset X$ is open and $V \cap Z \neq \emptyset$, then $\mathcal{O}_{X,Z} = \mathcal{O}_{V,V\cap Z}$.

(ii) Each $\mathcal{O}_{X,Z}$ is a *local ring* with maximal ideal:

$$m_Z := \{\phi \in \mathcal{O}_{X,Z} \mid \phi(Z) \equiv 0 \text{ (where defined)}\}$$

(iii) For each $\Phi : X \to Y$, let $W = \overline{\Phi(Z)} \subseteq Y$. Then there is a pull-back:

$$\Phi^* : \mathcal{O}_{Y,W} \to \mathcal{O}_{X,Z} \text{ with } \Phi^* m_W \subseteq m_Z$$

(iv) If X is affine, let $I(Z) \subset \mathbf{C}[X]$ be the prime ideal of Z. Then:

$$\mathcal{O}_{X,Z} = \mathbf{C}[X]_{I(Z)} \subset \mathbf{C}(X)$$

Thus by the correspondence between prime ideals $\mathbf{C}[X]_{I(Z)}$ and prime ideals in $\mathbf{C}[X]$ contained in $I(Z)$ (§7) there is an inclusion-reversing bijection:

$$\{\text{prime ideals } \mathcal{P} \subset \mathcal{O}_{X,Z}\} \leftrightarrow \{\text{closed subvarieties } W \text{ such that } Z \subseteq W \subseteq X\}$$

and in particular $0 \leftrightarrow X$ and $m_Z \leftrightarrow Z$.

So if $\mathrm{cod}_X(Z) = 1$, then $m_Z \subset \mathcal{O}_{X,Z}$ is the unique (non-zero) prime ideal!

As an example of rings with one non-zero prime ideal, consider:

Definition: If K is a field, a function $\nu : K^* \to \mathbf{Z}$ is a *discrete valuation* if:

(i) $\nu(ab) = \nu(a) + \nu(b)$ and

(ii) $\nu(a + b) \geq \min(\nu(a), \nu(b))$

for all $a, b \in K^*$. If ν is not the trivial (zero) valuation, then:

$$A_\nu := \{a \in K^* \mid \nu(a) \geq 0\} \cup \{0\}$$

is the *discrete valuation ring* (or DVR) associated to the valuation ν.

Examples: (a) Fix a prime p. For $a \in \mathbf{Z} - \{0\}$, define

$$\nu(a) = \text{the largest power of } p \text{ dividing } a$$

and $\nu(\frac{a}{b}) = \nu(a) - \nu(b)$ for $\frac{a}{b} \in \mathbf{Q}^*$. This is a discrete valuation and $A_\nu = \mathbf{Z}_{\langle p \rangle}$

(b) Fix a complex number $c \in \mathbf{C}$, and for non-zero $\phi \in \mathbf{C}(x)$ set:

$$\nu(\phi) = \text{the order of zero or pole (counted negatively) of } \phi \text{ at } x$$

This is a discrete valuation and $A_\nu = \mathbf{C}[x]_{\langle x-c \rangle}$

(c) Again, for non-zero rational functions $\phi = \frac{f}{g} \in \mathbf{C}(x)$, let:

$$\nu(\phi) = \deg(g) - \deg(f)$$

This is a discrete valuation and $A_\nu = \mathbf{C}[x^{-1}]_{\langle x^{-1} \rangle}$.

Observations: (a) If a valuation isn't surjective, its image is $d\mathbf{Z}$ for some $d > 0$, so we may as well divide through by d to get a surjective valuation.

(b) In a DVR A_ν (with surjective valuation ν), the (non-zero!) ideal:

$$m := \{a \in A_\nu \mid \nu(a) > 0\} \subset A_\nu$$

is principal, generated by any element $\pi \in A_\nu$ satisfying $\nu(\pi) = 1$. Such an element is called a *uniformizing parameter*. Note that $m = \langle \pi \rangle$ is the unique (non-zero) prime ideal since every ideal in A_ν is one of:

$$m^n = \langle \pi^n \rangle = \{a \in A \mid \nu(a) \geq n\}$$

for a uniquely determined n. Thus, in particular a DVR is always Noetherian and is a UFD (the factorization is $b = u\pi^n$ for unit u and unique n).

Proposition 11.1: Suppose A is a (local) Noetherian domain with a unique non-zero prime ideal $m \subset A$. Then:

(a) For any $f \in m$, the ring A_f is the field of fractions $K = K(A)$.

(b) If $I \subset A$ is any (non-zero) proper ideal, then $\sqrt{I} = m$.

(c) For each proper ideal $I \subset A$, there is a unique $n > 0$ such that:

$$m^n \subseteq I \text{ but } m^{n-1} \not\subseteq I$$

(d) If A is also integrally closed, then A is a DVR.

Proof: (a) A_f is a domain, and its prime ideals correspond to the prime ideals in A that do not contain f. Only the zero ideal has this property, so A_f has **no** non-zero primes, hence the zero ideal is maximal and $A_f = K$.

(b) Suppose $f \in m$ is arbitrary, and $b \in A$. Then $b^{-1} = \frac{a}{f^n} \in K$ by (a), so $ab = f^n$ for some n. In other words, $f \in \sqrt{\langle b \rangle}$ for all $f \in m$ and $b \in A$. So $m \subseteq \sqrt{I}$ for all ideals I and then $m = \sqrt{I}$ unless $\sqrt{I} = I = A$.

(c) It suffices to show that $m^n \subseteq I$ for some n. But this follows from (b). That is, if $\langle a_1, ..., a_k \rangle = m = \sqrt{I}$ (A is Noetherian!), then there are $n_1, ..., n_k$ such that each $a_i^{n_i} \in I$, and then we may let $n = 1 + \sum(n_i - 1)$ to ensure that $m^n \subseteq I$.

(d) Assume A is integrally closed. We'll first prove that m is principal. Pick an $a \in m$, and find n from (c) so that $m^n \subseteq \langle a \rangle$ but $m^{n-1} \not\subseteq \langle a \rangle$. Now choose $b \in m^{n-1} - \langle a \rangle$ and consider $\phi = \frac{b}{a} \in K$ (the field of fractions of A). By the choice of a and b, we know that $\phi \notin A$. On the other hand, $\phi m \subseteq A$ since $b \in m^{n-1}$ and $m^n \subseteq \langle a \rangle$. But $\phi m \not\subseteq m$, since if it were, we'd have:

$$A[\phi] \hookrightarrow m[\phi] = m; \ \sum \alpha_i \phi^i \mapsto \sum a\alpha_i \phi^i$$

an inclusion of A-modules and since A is integrally closed, $A[\phi]$ is not finitely generated, so this would violate Noetherianness of A. Thus there is a unit $u \in A$ and $\pi \in m$ such that $\phi\pi = u$. But now for any $a \in m$, we have $\phi a = a' \in A$, so $a = a'\phi^{-1} = (a'u^{-1})\pi$. So $m = \langle \pi \rangle$.

Now let's define the valuation. If $a \in A$, then either a is a unit or $a \in m$. In the latter case, we can write $a = \pi a_1$, and repeat the question of a_1. This gives an ascending chain of ideals:

$$\langle a \rangle \subset \langle a_1 \rangle \subset \langle a_2 \rangle \subset ...$$

that eventually stabilizes since A is Noetherian.

But this can only happen if $\langle a_n \rangle = A$ (since $\pi^{-1} \notin A$). Thus a_n is a unit, and we can write:

$$a = \pi^n a_n = u\pi^n$$

and the value of n in this expression is evidently unique. Thus we may define: $\nu(u\pi^n) = n$ and extend to a valuation on K^* with the desired $A = A_\nu$.

Proposition 11.2: A normal variety is non-singular in codimension 1.

Proof: Since the non-singular points of any variety are open (Prop 10.1) it suffices to show that given a normal variety X, there is no codimension 1 closed subvariety $Z \subset X$ such that $Z \subseteq \mathrm{Sing}(X)$.

For this, we may assume X is affine, so $\mathbf{C}[X]$ is integrally closed and then for any closed subvariety $Z \subset X$:

$$\mathcal{O}_{X,Z} = \mathbf{C}[X]_{I(Z)} \subset \mathbf{C}(X)$$

is also integrally closed (see Proposition 9.4). When Z has codimension 1, this is a Noetherian ring with a unique prime, so by Proposition 11.1, the ring $\mathcal{O}_{X,Z}$ is a DVR. In that case, let $\pi \in m_Z$ be a uniformizing parameter. Then $\pi \in \mathcal{O}_X(U)$ for some $U \cap Z \neq \emptyset$, and by shrinking U if necessary, we obtain an open subset $V \subset X$ such that:

- $V = U_f \subset X$ is a basic open set (hence an affine variety)
- $V \cap Z \neq \emptyset$
- $I(V \cap Z) = \langle \pi \rangle \subset \mathbf{C}[V]$

So let's replace X by V and Z by $Z \cap V$. Here's the main point. If $z \in Z$ and the images of $f_1, ..., f_m \in \mathbf{C}[X]$ under the map $\mathbf{C}[X] \to \mathbf{C}[Z] \to \mathcal{O}_{Z,z}$ are a basis for the Zariski cotangent space $T_z^*Z = m_z/m_z^2$ of Z at z, then because $I(Z) = \langle \pi \rangle$ it follows that $\mathcal{O}_{Z,z} = \mathcal{O}_{X,z}/\pi$ and so the images of $f_1, ..., f_m, \pi$ under $\mathbf{C}[X] \to \mathcal{O}_{X,z}$ also span the Zariski tangent space T_z^*X of X at z.

In other words, the dimensions of the Zariski tangent spaces satisfy:

$$\dim(\mathcal{O}_{X,z}) \leq \dim(\mathcal{O}_{Z,z}) + 1$$

But if $z \in Z$ is a non-singular point of Z, then $\dim(\mathcal{O}_{Z,z}) = \dim(Z)$ and then the inequality above is an equality since $\dim(X) = \dim(Z) + 1$ and $z \in X$ is also a non-singular point of X. Since there is a nonempty open set of nonsingular points in Z, it follows that $Z \not\subset \mathrm{Sing}(X)$, as desired.

Proposition 11.3: If $\Phi : X - - > \mathbf{CP}^n$ is a rational map and X is normal, then Φ is regular in codimension 1.

Proof: Recall that if $X \subset \mathbf{CP}^m$ then in a neighborhood of of each point $p \in X$ in the domain of X we can write $\Phi = (F_0 : .... : F_n)$ for some d and $F_i \in \mathbf{C}[x_0, ..., x_m]_d$ and conversely, so in particular, the domain of Φ is open. Thus to prove Φ is regular in codimension 1, we only have to prove that Φ is regular at *some* point $p \in Z$ of each codimension 1 closed subvariety $Z \subset X$.

By Proposition 11.1 each $\mathcal{O}_{X,Z} = \mathbf{C}(X)_\nu$ for some valuation ν on $\mathbf{C}(X)^*$. Consider now the valuations of the ratios:

$$\nu(\frac{F_i}{F_j}) \in \mathbf{C}(X) \text{ for some expression } \Phi = (F_0 : .... : F_n)$$

valid near some point of X. If we choose $\nu(\frac{F_{i_0}}{F_{j_0}})$ minimal (certainly negative!), then since they satisfy: $\nu(\frac{F_{i_0}}{F_{j_0}}) + \nu(\frac{F_i}{F_{i_0}}) = \nu(\frac{F_i}{F_{j_0}}) \geq \nu(\frac{F_{i_0}}{F_{j_0}})$, each $\nu(\frac{F_i}{F_{i_0}}) \geq 0$. Thus the components of the rational map:

$$\Phi : X - - > \mathbf{C}^n \cong U_{i_0} \subseteq \mathbf{CP}^n; p \mapsto (\frac{F_0}{F_{i_0}}, ..., \frac{F_n}{F_{i_0}})$$

are all in $\mathcal{O}_{X,Z}$, so for some open subset $U \subset X$ with $U \cap Z \neq \emptyset$, each component belongs to U, and then Φ is regular all along U, as desired.

Example: Consider the map to the line through the origin:

$$\pi : \mathbf{C}^2 - - > \mathbf{CP}^1;\ (x, y) \mapsto (x : y)$$

As we've seen, this cannot be extended to a regular map across $(0, 0)$. But if we restrict this to the (nonsingular) curve $C = V(y^2 - x(x - 1)(x - \lambda))$:

$$\pi|_C : C \to \mathbf{CP}^1$$

becomes regular across $(0, 0)$. That's because (as in the Proposition):

$$x = \frac{y^2}{(x - 1)(x - \lambda)}$$

in a neighborhood of $(0, 0)$, and then in that neighborhood:

$$\pi(x, y) = (\frac{y^2}{(x - 1)(x - \lambda)} : y) = (\frac{y}{(x - 1)(x - \lambda)} : 1)$$

so $\pi(0, 0) = (0 : 1)$ (the vertical tangent line!). Notice that we couldn't do this if $\lambda = 0$, but in that case C is singular at $(0, 0)$.

Remark: Putting Propositions 11.2 and 11.3 together in dimension 1 gives a very striking result. Namely, if $X \subset \mathbf{CP}^n$ is any quasi-projective variety of dimension 1 (i.e. a curve), let C be the normalization of the closure of X. Then C is a non-singular (Proposition 11.2) projective curve with:

$$\mathbf{C}(C) \cong \mathbf{C}(X)$$

Moreover, if C' is any other non-singular projective curve with:

$$\mathbf{C}(C') \cong \mathbf{C}(C)$$

then the associated birational map:

$$\Phi : C - - > C'$$

is an isomorphism(!) (by Proposition 11.3 applied to Φ and Φ^{-1}). Thus up to isomorphism, there is only one non-singular projective curve with a given field of rational functions. Moreover, if: $\mathbf{C}(C) \subset \mathbf{C}(C'')$ is an inclusion of such fields, then again by Proposition 11.3, the associated rational map:

$$\Phi : C'' \to C$$

is actually regular, so the field inclusions correspond to (finite) maps of the non-singular projective "models" of the fields. We will study more properties of non-singular projective curves later.

Proposition 11.4: If $\Phi : X \to Y$ is a birational surjective regular map and Y is non-singular, then if Φ is not an isomorphism, it is also not an isomorphism in codimension 1.

Proof: If Φ is not an isomorphism, let $q \in Y$ be a point where Φ^{-1} is not defined and let $p \in \Phi^{-1}(q)$ (if Φ^{-1} were defined everywhere, then it would be a regular inverse of Φ!). Consider affine neighborhoods $q \in V$ and $p \in U \subset \Phi^{-1}(V)$, and let $U \subset \mathbf{C}^n$ be a closed embedding. Then locally:

$$\Phi^{-1} = (\phi_1, ..., \phi_n) : V - - > U \subset \mathbf{C}^n$$

and to say that $\Phi^{-1}(q)$ is not defined is to say that some $\phi = \phi_i \notin \mathcal{O}_{Y,q}$. But we've proved that $\mathcal{O}_{Y,q}$ is a UFD in §10, so we can write $\phi = \frac{f}{g}$ in lowest terms, with $f, g \in \mathcal{O}_{Y,q}$. When we pull back: $\Phi^*(\phi) = x_i \in \mathbf{C}[U]$ is the ith coordinate function, so:

$$\Phi^*(f) = x_i \Phi^*(g) \in \mathcal{O}_{X,p}$$

and we can shrink V (and U) further so that $f, g \in \mathbf{C}[V]$.

Consider the hypersurface $V(\Phi^*(g)) \subset U$. We know $p \in V(\Phi^*(g))$ since otherwise $g(q) \neq 0$ would give $\frac{f}{g} \in \mathcal{O}_{Y,q}$. Each component $Z \subset V(\Phi^*(g))$ (containing p) has codimension 1 in U by Krull's Principal Ideal Theorem. On the other hand, the image:

$$\Phi(Z) \subset V(f) \cap V(g)$$

must have codimension ≥ 2 in V. That's because if we write:

$$g = ug_1 \cdots g_n$$

as a product of irreducibles (and shrink V again so that each $g_i \in \mathbf{C}[V]$) then the $V(g_i)$ are the irreducible components of $V(g)$ and $f \notin (g_i) = I(V(g_i))$ so $V(f) \cap V(g_i) \neq V(g_i)$, and $V(Z)$ is contained in one of these.

Finally, by the theorem on the dimension of the fibers of a regular map, we see that since $\dim(Z) > \dim(\Phi(Z))$, it follows that every $z \in Z$ is contained in a fiber of Φ of dimension ≥ 1, so Φ is not an isomorphism anywhere on Z.

We've actually shown that the set of points $p \in X$ where Φ fails to be an isomorphism is a union of codimension 1 subvarieties of X. This is called the *exceptional locus* of the birational map Φ.

Moral Example: It's time you saw the *Cremona transformation*:

$$\Phi : \mathbf{CP}^2 - - > \mathbf{CP}^2; (x : y : z) \mapsto (\frac{1}{x} : \frac{1}{y} : \frac{1}{x}) = (yz : xz : xy)$$

By Proposition 11.3, this is regular in codimension 1, and indeed it is:

$$\text{domain}(\Phi) = \mathbf{CP}^2 - \{(0:0:1), (0:1:0), (1:0:0)\}$$

Note that $\Phi^2 = \text{id}$, but the exceptional locus of Φ are $V(x) \cup V(y) \cup V(z)$ (this is only morally an example, since Φ isn't regular and it isn't surjective).

Main Example: If X is non-singular, then any rational map:

$$\Phi : X - - > \mathbf{CP}^n$$

gives rise to a birational, surjective regular projection $\sigma = \pi_1 : \overline{\Gamma}_\Phi \to X$ from the closure of the graph of Φ (in $X \times \mathbf{CP}^n$) to X. Evidently, σ restricts to an isomorphism from Γ_Φ to the domain of Φ, and the Proposition tells us that the exceptional set of σ is a union of codimsion 1 subvarieties, which is to say that the "boundary" $\overline{\Gamma}_\Phi - \Gamma_\Phi$ is a union of codimension 1 subvarieties mapping to the "indeterminate locus" $X - \text{dom}(\Phi)$ of the rational map Φ.

The simplest instance of this is:

The Blow up of a point in $\mathbf{C}^n$: Consider the rational map:

$$\Phi : \mathbf{C}^n - - > \mathbf{CP}^{n-1}; \quad \Phi(p_1, ..., p_n) = (p_1 : ... : p_n)$$

In other words, Φ maps a point $p \in \mathbf{C}^n$ (other than the origin) to the point in projective space corresponding to the line through p and the origin. Then:

$$bl_0(\mathbf{C}^n) := \overline{\Gamma}_\Phi = \Gamma_\Phi \cup (\{0\} \times \mathbf{CP}^{n-1})$$

because $\sigma^{-1}(0) \subseteq (\{0\} \times \mathbf{CP}^{n-1})$ is required to have codimension 1 in $bl_0(\mathbf{C}^n)$ (which has dimension n). This exceptional variety "nicely" fits with Γ_Φ:

Proposition 11.5: $bl_0(\mathbf{C}^n) \subset \mathbf{C}^n \times \mathbf{CP}^{n-1}$ is:

(a) non-singular and

(b) given by explicit equations (which we give in the proof).

(σ is sometimes called a blow-down, σ-process or monoidal transformation)

Proof: Consider the ring $\mathbf{C}[x_1, ..., x_n, y_1, ..., y_n]$ graded by degree in y, as in the proof of Proposition 6.7. We've seen that the closed subsets of $\mathbf{C}^n \times \mathbf{CP}^{n-1}$ are all of the form $V(I)$ for homogeneous ideals I. Then:

$$bl_0(\mathbf{C}^n) = V := V(\langle x_i y_j - x_j y_i \rangle)$$

since this contains Γ_Φ (and no other points over $0 \neq p \in \mathbf{C}^n$) and is closed and irreducible. Note that V does contain $0 \times \mathbf{CP}^{n-1}$, as we proved it must(!) We see that V is irreducible and nonsingular by covering it with open affine subsets. Namely, let $U_i = \mathbf{C}^n \times (\mathbf{CP}^{n-1} - V(y_i))$. Then:

$$\mathbf{C}[U_i]/I(V \cap U_i) \cong \mathbf{C}[x_1, ..., x_n, \frac{y_1}{y_i}, ..., \frac{y_n}{y_i}]/\langle x_i(\frac{y_j}{y_i}) - x_j, ..., x_k(\frac{y_j}{y_i}) - x_j(\frac{y_k}{y_i})\rangle$$

but all relations follow from the first ones, solving $x_j = x_i \frac{y_j}{y_i}$ so this ring is a domain, $V \cap U_i \subset U_i$ is closed and irreducible, and:

$$\mathbf{C}[U_i]/I(V \cap U_i) = \mathbf{C}[V \cap U_i] \cong \mathbf{C}[x_i, \frac{y_1}{y_i}, ..., \frac{y_n}{y_i}]$$

so that $bl_0(\mathbf{C}^n) \cap U_i = V \cap U_i \cong \mathbf{C}^n$ is, in particular, nonsingular!

Remark: $bl_0(\mathbf{C}^n)$ defined above is not an affine variety, since there is a closed embedding $\mathbf{CP}^{n-1} \hookrightarrow bl_0(\mathbf{C}^n)$. It is quasi-projective, of course, as it sits inside $\mathbf{C}^n \times \mathbf{CP}^{n-1}$ as a (non-singular) closed subvariety.

First Generalization: Blowing up a point in $\mathbf{CP}^n$. Consider

$$\pi : \mathbf{CP}^n - - > \mathbf{CP}^{n-1}$$

the projection from a point $p \in \mathbf{CP}^n$. With suitable basis, we can assume $p = 0 \in \mathbf{C}^n \subset \mathbf{CP}^n$. It follows that:

$$bl_p(\mathbf{CP}^n) := \overline{\Gamma}_\pi \subset \mathbf{CP}^n \times \mathbf{CP}^{n-1}$$

is a non-singular projective variety, with open (not affine!) cover given by:

$$bl_p(\mathbf{CP}^n) = bl_0(\mathbf{C}^n) \cup (\mathbf{CP}^n - p)$$

and these open sets patch by the projection $\sigma : \Gamma_\pi \overset{\sim}{\to} \mathbf{CP}^n - p$.

That was the easy generalization. Next:

Proposition 11.6: If $X \subset \mathbf{CP}^n$ is an embedded variety (maybe singular) of dimension m and $p \in X$ is a (maybe singular) point, then the variety:

$$bl_p(X) := \overline{\Gamma}_{\pi|_X} = \Gamma_{\pi|_X} \cup PC_pX \subset bl_p(\mathbf{CP}^n)$$

where $PC_pX \subset \{p\} \times \mathbf{CP}^{n-1}$ is the "projectivized tangent cone" of X at p. If p is nonsingular then $PC_p(X) \subset \mathbf{CP}^{n-1}$ is naturally identified with the projective tangent space to X at p, and $bl_p(X)$ is nonsingular along $PC_p(X)$.

Proof: Let $Y = X \cap \mathbf{C}^n$ for $p = 0 \in \mathbf{C}^n \subset \mathbf{CP}^n$ as above. It follows from Proposition 11.5 that the blow-up of Y at p is the unique component:

$$bl_p(Y) \subset V(\langle x_iy_j - x_jy_i, f(x_1, ..., x_n)) \mid f \in I(Y)\rangle)$$

containing the graph of $\pi|_Y : Y - - > \mathbf{CP}^{n-1}$. This is covered by:

$$bl_p(Y) \cap U_i \subset V(\langle f(x_i\frac{y_1}{y_i}, ..., x_i\frac{y_n}{y_i})\rangle) \subset \mathbf{C}^n = bl_p(\mathbf{C}^n) \cap U_i$$

Expand f as a sum of homogeneous polynomials $f = f_{d_0} + f_{d_0+1} + ...f_d$. with $f_{d_0} \neq 0$. Since $0 \in Y$ we have $d_0 \geq 1$ and then:

$$f(x_i\frac{y_1}{y_i}, ..., x_i\frac{y_n}{y_i}) = x_i^{d_0}(f_{d_0}(\frac{y_1}{y_i}, ..., \frac{y_n}{y_i}) + x_if_{d_0}(\frac{y_1}{y_i}, ..., \frac{y_n}{y_i}) + ...)$$

gives:

$$V(\langle f(x_i\frac{y_1}{y_i}, ..., x_i\frac{x_n}{x_i})\rangle) = V(x_i) \cup V(\langle \frac{f(x_i\frac{y_1}{y_i}, ..., x_i\frac{y_n}{y_i})}{x_i^{d_0}}\rangle)$$

Consider the inclusion of polynomial rings (with $z_j = \frac{y_j}{y_i}$ for convenience):

$$A = \mathbf{C}[x_iz_1, ..., x_iz_{i-1}, x_i, x_iz_{i+1}, ..., x_iz_n] \subset \mathbf{C}[z_1, ..., z_{i-1}, x_i, z_{i+1}, ..., z_n] = B$$

The ideal $P = \{f(x_iz_1, ..., x_iz_n) \mid f \in I(Y)\} \subset A$ is prime by assumption, and the ideal $Q = \langle \frac{1}{x_i^{d_0}} f(x_iz_1, ..., x_iz_n) \mid f \in I(Y)\rangle \subset B$ is equal to $P_{x_i} \cap B$ for the natural inclusion $B \subset A_{x_i}$. So it is prime, too.

Since $V(x_i) = \pi_1^{-1}(0) \subset \mathbf{C}^n \times \mathbf{CP}^{n-1}$ and $V(Q)$ is irreducible, we get:

$$bl_p(Y) \cap U_i = V(Q) \;\text{ and }\; \sigma^{-1}(0) \cap U_i = V(Q) \cap V(x_i) = V(\langle f_{d_0}(\frac{y_1}{y_i}, ..., \frac{y_n}{y_i})\rangle)$$

Thus $\sigma^{-1}(0) = V(\langle f_{d_0}(y_1, ..., y_n)\rangle) =: PC_pX \subset \mathbf{CP}^{n-1}$ is the projectivized tangent cone to Y (or X) at p. To repeat, the $f_{d_0}(x_1, ..., x_n)$ are the leading terms in the Taylor polynomials at p of the equations of $Y \subset \mathbf{C}^n$.

For the last part, notice that if $F \in I(X)$ and $f = F(1, x_1, ..., x_n) \in I(Y)$ then

$$f_1 = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)x_i = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(p)x_i \text{ (up to a scalar)}$$

But

$$V(\sum_{i=0}^n \frac{\partial F}{\partial x_i}(p)x_i) = \pi(V(\sum_{i=0}^n \frac{\partial F}{\partial x_i}(p)x_i))$$

from which it follows that $PC_p(X) \subseteq \pi(\Theta_p)$ (see Exercise 10.5). But if p is a nonsingular point, then $\pi(\Theta_p)) \cong \mathbf{CP}^{m-1}$ so $PC_p(X) = \pi(\Theta_p)$ since they have the same dimension! Finally, if p is non-singular, then:

$$\langle x_i\rangle = \langle x_1, ..., x_n, f_1(\frac{y_1}{y_i}, ..., \frac{y_n}{y_i}) \mid f \in I(Y)\rangle = I(PC_p(X) \cap U_i)$$

so the ideal of the non-singular subvariety $PC_p(X) \cap U_i \subset bl_p(Y) \cap U_i$ is generated by x_i, and then as in the proof of Proposition 11.1, it follows that $bl_p(Y)$ is non-singular along $PC_p(X)$.

Examples: (a) If $Y = V(y^2 - x^2 + x^3) \subset \mathbf{C}^2$ is the nodal cubic, then:

$$bl_0(Y) = V(y - x\frac{y_2}{y_1}, \left(\frac{y_2}{y_1}\right)^2 - 1 + x) \cup V(x - y\frac{y_1}{y_2}, 1 - \left(\frac{y_1}{y_2}\right)^2 + y\left(\frac{y_1}{y_2}\right)^3)$$

is non-singular (the first open set is a parabola!), and:

$PC_p(Y) = V(y^2 - x^2) = (0:1) \cup (1:0) \subset \mathbf{CP}^1$ consists of two points

(b) If $Y = V(y^2 - x^5)$, then:

$$bl_0(Y) = V(y - x\frac{y_2}{y_1}, \left(\frac{y_2}{y_1}\right)^2 - x^3) \cup V(x - y\frac{y_1}{y_2}, 1 - y^3\left(\frac{y_1}{y_2}\right)^5)$$

and this is singular at the exceptional locus:

$$PC_p(Y) = V(y^2) = (1:0) \in \mathbf{CP}^1$$

Notice that $PC_p(Y)$ is a single point (evidently nonsingular) but $bl_0(Y)$ is singular at this point. This is no contradiction since the *ideal* of the point is not given by a single equation!

Final Remark: We can view the blow-up $bl_p(X)$ as a surgery, removing the point $p \in X$ and replacing it with the codimension 1 projectivized tangent cone $PC_p(X) \subset bl_p(X)$. We can ask whether blowing up X along an *isolated* singular point $p \in X$ "improves" X, in the sense that $bl_p(X)$ is less singular than X was. The situation is somewhat complicated in dimension ≥ 3, but in dimension 2, the answer is yes. That is, if we start with an arbitrary projective surface X and normalize, then by Proposition 11.2, the resulting normal surface S has only finitely many (isolated) singularities, and then blowing up these singularities (and any singularities that occur in the exceptional loci) will eventually produce a non-singular projective surface S_n:

$$S_n = bl_{p_n}(S_{n-1}) \to S_{n-1} = bl_{p_{n-1}}(S_{n-2}) \to S_1 = \cdots bl_{p_1}(S) \to S$$

Similarly, if $\Phi : S - - > S'$ is a rational map of normal projective surfaces, then by Proposition 11.3, the indeterminacy locus of Φ is a finite set of points, and it is also true that after finitely many blow-ups, we obtain a regular map:

$$\begin{array}{ccc}
S_n = bl_{p_n}(S_{n-1}) & & \\
\downarrow & & \\
\vdots & & \\
\downarrow & \searrow & \\
S_1 = bl_{p_1}(S) & & \\
\downarrow & & \\
\Phi : S & - - > & S'
\end{array}$$

Thus with the assistance of blow-ups at points, the situation is somewhat similar (but more complicated) than the situation for curves.

Exercises 11.

1. If X is a normal variety, let:

$$\mathrm{Div}(X) = \bigoplus_{\text{cod 1 irred } V \subset X} \mathbf{Z}[Z]$$

i.e. $\mathrm{Div}(X)$ is the free abelian group generated by the codimension 1 closed subvarieties of X. Prove that there is a well-defined homomorphism:

$$\mathrm{div} : \mathbf{C}(X)^* \to \mathrm{Div}(X);\ \mathrm{div}(\phi) = \sum_V \nu_Z(\phi)$$

where ν_Z is the discrete valuation defining $\mathcal{O}_{X,Z}$. Identify the kernel of div and prove that if X is an affine variety, then:

$$\mathbf{C}[X] \text{ is a UFD } \Leftrightarrow \text{ div is surjective}$$

(the quotient $\mathrm{Div}(X)/\mathrm{div}(\mathbf{C}(X)^*)$ is called the "class group" of X).

2. Consider the projection from $p = (1 : 0 : 0 : 0)$:

$$\pi : \mathbf{CP}^1 \times \mathbf{CP}^1 = V(xw - yz) - - > \mathbf{CP}^2;\ (a : b : c : d) \mapsto (b : c : d)$$

Prove that $bl_p(\mathbf{CP}^1 \times \mathbf{CP}^1)$ is isomorphic to the "compound" blowup:

$$X := bl_{\sigma^{-1}(0:1:0)}(bl_{(1:0:0)}(\mathbf{CP}^2))$$

(this is the blow-up of $\mathbf{CP}^2$ at the two points $(0 : 1 : 0)$ and $(1 : 0 : 0)$).

3. If $X = \overline{C(V)} \subset \mathbf{PC}^n$ is the projective cone over $V \subset \mathbf{CP}^{n-1}$, show that

$$PC_p(X) = V$$

where $p = 0 \in \mathbf{C}^n \subset \mathbf{CP}^n$ is the vertex of the cone and show that:

$$\pi_2 : bl_p(X) \to \mathbf{CP}^{n-1}$$

has fibers isomorphic to $\mathbf{CP}^1$. In fact, find isomorphisms:

$$bl_p(X) \cap U_i \cong \mathbf{CP}^1 \times (X - V(x_i))$$