Math 6130 Notes. Fall 2002.

3. Affine Varieties. These are geometric objects associated to the domains:

$$\mathbf{C}[x_1, ..., x_n]/P$$

that define "local" complex algebraic geometry.

Definition: (a) A subset $V \subseteq \mathbf{C}^n$ is an *algebraic set* if there is an ideal $I \subseteq \mathbf{C}[x_1, ..., x_n]$ for which $V = V(I)$ (see Corollary 1.4).

(b) An ideal $I \subseteq \mathbf{C}[x_1, ..., x_n]$ is *radical* if $I = \sqrt{I}$ (see Corollary 1.4).

Proposition 3.1: (a) Every algebraic set $V \subseteq \mathbf{C}^n$ is the zero locus:

$$V = \{f_1(x_1, ..., x_n) = f_2(x_1, ..., x_n) = ... = f_m(x_1, ..., x_n) = 0\}$$

of a finite set of polynomials $f_1, ..., f_m \in \mathbf{C}[x_1, ..., x_n]$.

(b) The maps $V \mapsto I(V)$ and $I \mapsto V(I)$ of Corollary 1.4 give a bijection:

$$\{\text{algebraic subsets } V \subseteq \mathbf{C}^n\} \leftrightarrow \{\text{radical ideals } I \subseteq \mathbf{C}[x_1, ..., x_n]\}$$

(c) A topology on $\mathbf{C}^n$, called the *Zariski topology*, results when:

$$U \subseteq \mathbf{C}^n \text{ is open } \Leftrightarrow Z := \mathbf{C}^n - U \text{ is an algebraic set}$$

Proof: For (a), note that if $I = \langle f_1, ..., f_m \rangle$, then

$$V(I) = \{f_1(x_1, ..., x_n) = f_2(x_1, ..., x_n) = ... = f_m(x_1, ..., x_n) = 0\}$$

and every ideal is generated by finitely many f_i by the Basis Theorem.

For (b), recall that Corollary 1.4 of the Nullstellensatz tells us that:

$$I(V(I)) = \sqrt{I}$$

so $I(V(I)) = I$ if I is a radical ideal. Since $V(I) = V(\sqrt{I})$ and $\sqrt{\sqrt{I}} = \sqrt{I}$, it follows that any algebraic set is $V(I)$ for a radical ideal I, hence the bijection.

Finally, for (c), we need the following properties, which are easily checked:

(i) $\emptyset = V(\langle 1 \rangle)$ and $\mathbf{C}^n = V(\langle 0 \rangle)$ are closed.

(ii) If $V_1 = V(I_1)$ and $V_2 = V(I_2)$, then $V_1 \cup V_2 = V(I_1 I_2)$ is closed.

(iii) If $V_\lambda = V(I_\lambda)$ for $\lambda \in \Lambda$, then $\cap_{\lambda \in \Lambda} V_\lambda = V(\sum_{\lambda \in \Lambda} I_\lambda)$ is closed.

Remark: The bijection of (b) is "inclusion reversing," i.e.

$$V_1 \subseteq V_2 \Leftrightarrow I(V_1) \supseteq I(V_2)$$

as is easily checked from the definition.

Definition: A closed set $Z \subseteq \mathbf{C}^n$ is *irreducible* if there is no pair of proper closed subsets $Z_1, Z_2 \subset Z$ with the property that $Z = Z_1 \cup Z_2$.

Proposition 3.2: Features of the Zariski topology include the following:

(a) Every descending chain of closed sets: $\mathbf{C}^n \supseteq Z_1 \supseteq Z_2 \supseteq ...$ stabilizes.

(b) Every non-empty open subset $U \subseteq \mathbf{C}^n$ is dense.

(c) In the correspondence of Proposition 3.1(b), the irreducible algebraic sets correspond to prime ideals.

(d) Every closed set $Z \subseteq \mathbf{C}^n$ is a finite union of irreducible closed subsets.

Proof: Part (a) follows from the bijection of Proposition 3.1(b), since ascending chains of radical ideals in $\mathbf{C}[x_1, ..., x_n]$ stabilize (Proposition 1.1(a)).

Part (b) is just a restatement of the assertion that $\mathbf{C}^n$ itself is irreducible (if U weren't dense, then $\overline{U} \cup (\mathbf{C}^n - U) = \mathbf{C}^n$ so $\mathbf{C}^n$ wouldn't be irreducible, and if $\mathbf{C}^n = Z_1 \cup Z_2$, then $U = \mathbf{C}^n - Z_1 \subset Z_2$ wouldn't be dense!) Since $\mathbf{C}^n$ corresponds to the zero ideal under the bijection of Proposition 3.1(b), this will follow from (c).

As for (c), suppose Z is a closed set. If $I := I(Z)$ isn't prime, then there are $f, g \notin I$ such that $fg \in I$, and then $Z_1 := V(\langle f \rangle + I)$ and $Z_2 := V(\langle g \rangle + I)$ satisfy: $Z_1 \cup Z_2 = V((\langle f \rangle + I)(\langle g \rangle + I)) = Z$ showing that Z isn't irreducible. On the other hand, if Z isn't irreducible, then write $Z = Z_1 \cup Z_2$ and let $f \in I(Z_1) - I$, $g \in I(Z_2) - I$, and then $fg \in I$ shows that I isn't prime.

Finally, if some closed set $Z \subseteq \mathbf{C}^n$ were not a finite union of irreducible closed subsets, we could express $Z = Z_1 \cup Z_1'$ as a union of proper closed sets such that one (or both) of them, say Z_1, is also not a finite union of irreducibles. But then $Z_1 = Z_2 \cup Z_2'$ decomposes in the same way, and we obtain inductively a chain of closed sets:

$$\mathbf{C}^n \supset Z \supset Z_1 \supset Z_2 \supset ...$$

that never stabilizes, in violation of (a).

Remark: The irreducible closed subsets Z_i in the finite union from (d):

$$Z = Z_1 \cup ... \cup Z_m$$

are unique (up to permutation) and are the *(irreducible) components* of Z.

Example: A *hypersurface* is the zero locus of one non-zero polynomial:

$$V(f) = \{(a_1, ..., a_n) \in \mathbf{C}^n \mid f(a_1, ..., a_n) = 0\} \subset \mathbf{C}^n$$

By Proposition 3.1(a), hypersurfaces generate all the closed sets in the Zariski topology, since $V(I) = V(f_1) \cap ... \cap V(f_m)$ when $I = \langle f_1, ..., f_m \rangle$. Equivalently, all open sets $U \subset \mathbf{C}^n$ are finite unions of *basic open sets*, which are defined to be the complements $\mathbf{C}^n - V(f)$ of hypersurfaces.

Since $\mathbf{C}[x_1, ..., x_n]$ is a UFD, the principal ideal $\langle f \rangle$ decomposes as a product of prime ideals according to the factorization: $f = \prod_{i=1}^m f_i$ as a product of irreducible polynomials. If no f_i appears with multiplicity > 1, then $\langle f \rangle$ is radical, so $\langle f \rangle = I(V(f))$ and:

$$V(f) = V(f_1) \cup V(f_1) \cup ... \cup V(f_m)$$

is the expression of $V(f)$ as a union of its irreducible components.

Definition: An *affine variety* is any irreducible closed subset $X \subseteq \mathbf{C}^n$. The *Zariski topology* on X is induced from the Zariski topology on $\mathbf{C}^n$ and the *hypersurfaces* and *basic open sets* in X are the (proper) intersections of X with hypersurfaces of $\mathbf{C}^n$ and their complements in X, respectively. The *affine coordinate ring of* X is the domain $\mathbf{C}[X] := \mathbf{C}[x_1, ..., x_n]/P$ where $P = I(X)$ is the prime ideal associated to X in Proposition 3.2(c).

Observation: The algebraic (i.e. closed) subsets of X are in bijection with the radical ideals of $\mathbf{C}[x_1, ..., x_n]$ containing P, i.e. the radical ideals of $\mathbf{C}[X]$. The irreducible closed sets in X are in bijection with the prime ideals in $\mathbf{C}[X]$.

Examples: (a) The hyperbolas $V(xy - a) \subset \mathbf{C}^2$ ($a \neq 0$) are affine varieties (even though they don't look irreducible when we visualize them in $\mathbf{R}^2$!). Recall from §1 that if $X = V(xy - a)$, then:

$$\mathbf{C}[X] = \mathbf{C}[x, y]/\langle xy - a \rangle \xrightarrow{\sim} \mathbf{C}[t, t^{-1}]; \ \overline{x} \mapsto t, \ \ \overline{y} \mapsto at^{-1}$$

and this ring is a principal ideal domain, with prime ideals $\langle t - b \rangle$ ($b \neq 0$). Thus the irreducible closed sets in X are the points $(b, ba^{-1}) \in X$, and all closed subsets (other than X itself) are finite.

(b) The cuspidal cubic $V(y^2 - x^3) \subset \mathbf{C}^2$ is also an affine variety, with:

$$\mathbf{C}[X] = \mathbf{C}[x,y]/\langle y^2 - x^3\rangle \xrightarrow{\sim} \mathbf{C}[t^2, t^3]; \;\; \overline{x} \mapsto t^2, \; \overline{y} \mapsto t^3$$

This is not a principal ideal domain, because in particular the maximal ideals $\langle t^2 - b^2, t^3 - b^3\rangle$ are not principal. But the points of X are still the only irreducible closed sets since any nonzero polynomial $f(t^2, t^3)$ vanishes for only finitely many values of t, so every closed subset of X is finite.

(c) The nodal cubic $V(y^2 - x^2(x+1)) \subset \mathbf{C}^2$ is also an affine variety, with

$$\mathbf{C}[X] = \mathbf{C}[x,y]/\langle y^2 - x^2(x+1)\rangle \xrightarrow{\sim} \mathbf{C}[t^2 - 1, t(t^2 - 1)]$$

which is once again not a principal ideal domain, but by the same argument as in (b), all hypersurfaces, hence all closed subsets are finite.

(d) Rather surprisingly, it is the "nonsingular" cubic:

$$X = V(y^2 - (x^3 - a))$$

whose Zariski topology is hardest to see, since $\mathbf{C}[X]$ is neither a PID nor a subring of $\mathbf{C}[t]$. The question is: Why are the hypersurfaces in X all finite?

(e) Suppose $f(x,y), g(x,y) \in \mathbf{C}[x,y]$ are relatively prime of degrees d, e. Then their *homogenizations* from §2 (changing variables $x \leftrightarrow \frac{x_1}{x_0}, y \leftrightarrow \frac{x_2}{x_0}$) $F(x_0, x_1, x_2) \in \mathbf{C}[x_0, x_1, x_2]_d$, $G(x_0, x_1, x_2) \in \mathbf{C}[x_0, x_1, x_2]_e$ are still relatively prime, and so in particular, $\overline{G} \in \mathbf{C}[x_0, x_1, x_2]/\langle F\rangle$ is not a zero-divisor.

Thus the Hilbert polynomial of $\mathbf{C}[x_0, x_1, x_2]/\langle F, G\rangle$ is the constant de, and it follows (see Exercise 2.4) that $V(F) \cap V(G) \subset \mathbf{CP}^2$ is finite. In fact:

$$|V(F) \cap V(G)| \le de$$

and so it follows also that $V(f) \cap V(g) \le de$. From this we see that:

- Every hypersurface in every "plane curve" affine variety $V(f) \subset \mathbf{C}^2$ is finite, so every closed subset is finite, and

- The irreducible closed sets in $\mathbf{C}^2$ are either irreducible hypersurfaces $V(f) \subset \mathbf{C}^2$ or points, since any irreducible closed subset is contained in an irreducible hypersurface. Thus the closed sets are finite unions of points and irreducible hypersurfaces.

Definition: A *regular function* on an affine variety X is any $\overline{f} \in \mathbf{C}[X]$.

Notice that a regular function is a well-defined function on X. That is,

$$\overline{f}(a_1, ..., a_n) = f(a_1, ..., a_n) \in \mathbf{C}$$

where $f \in \mathbf{C}[x_1, ..., x_n]$ is any polynomial reducing to $\overline{f}$ mod P. Since the polynomials in P vanish at all points of X by definition, this is well-defined.

Definition: A *rational function* on X is an element $\phi \in \mathbf{C}(X)$ of the field of fractions of the coordinate ring $\mathbf{C}[X]$.

A rational function ϕ is thus the ratio of two regular functions $\frac{\overline{f}}{\overline{g}}$. If $\mathbf{C}[X]$ is a UFD (as in the case of $\mathbf{C}^n$ itself) then the ratio can be put in lowest terms to get a preferred expression for ϕ. In general, however, there may be several equally good expressions for ϕ as ratios of regular functions. If $(a_1, ..., a_n) \in X$ and if there is (at least) one such expression for which $\overline{g}(a_1, ..., a_n) \neq 0$, then $\phi(a_1, ..., a_n)$ is well-defined, and ϕ is also a well-defined function in an open neighborhood of $(a_1, ..., a_n)$. In this case we say that ϕ is *regular* at $(a_1, ..., a_n)$, otherwise we say that ϕ has a *pole* at $(a_1, ..., a_n)$.

Proposition 3.3: The only rational functions $\phi \in \mathbf{C}(X)$ that are regular at all the points of X are the regular functions.

Proof: Consider all the possible expressions for such a ϕ as a ratio $\phi = \frac{\overline{f}}{\overline{g}}$. The set of denominators $\overline{g}$ that occur (and 0) is an ideal $I_\phi \subseteq \mathbf{C}[X]$ since:

$$\phi = \frac{\overline{f}_1}{\overline{g}_1} = \frac{\overline{f}_2}{\overline{g}_2} \Rightarrow \phi = \frac{\overline{f}_1 + \overline{f}_2}{\overline{g}_1 + \overline{g}_2} \text{ and } \phi = \frac{\overline{h}\,\overline{f}_1}{\overline{h}\,\overline{g}_1} \text{ unless } \overline{g}_1 + \overline{g}_2 = 0 \text{ or } \overline{h} = 0$$

By assumption, given $(a_1, ..., a_n) \in X$ there is a $\overline{g} \in I_\phi$ with $\overline{g}(a_1, ..., a_n) \neq 0$. But maximal ideals of $\mathbf{C}[X]$ are maximal ideals of $\mathbf{C}[x_1, ..., x_n]$ containing P, which, by the Nullstellensatz, are the ideals of the form $\langle x_1 - a_1, ..., x_n - a_n\rangle$ for $(a_1, ..., a_n) \in X$. So the assumption tells us that I_ϕ is contained in no maximal ideal, which is to say that $1 \in I_\phi$, so $\phi = \overline{f}$ for some $\overline{f}$, as desired.

Example: In $\mathbf{C}[x_1, x_2, x_3, x_4]/\langle x_1x_4 - x_2x_3\rangle$, the rational function $\phi = \frac{\overline{x}_1}{\overline{x}_2} = \frac{\overline{x}_3}{\overline{x}_4}$ is regular on $\{(a, b, c, d) \mid b \neq 0 \text{ or } d \neq 0 \text{ and } ad = bc\} \subset X = \{ad = bc\}$ meaning that the locus where ϕ has a pole is the plane:

$$\mathbf{C}^2 = \{(a, 0, c, 0)\} \subset X \subset \mathbf{C}^4$$

and ϕ does *not* have a pole at points $(0, 0, c, d)$ $(d \neq 0)$ and $(a, b, 0, 0)$ $(b \neq 0)$ even though one of the expressions for ϕ is $\frac{0}{0}$ at such points.

Definition: The *sheaf of regular functions* $\mathcal{O}_X$ on X is defined by:

$$\mathcal{O}_X(U) := \{\phi \in \mathbf{C}(X) \mid \phi \text{ is regular at all points of } U\}$$

for Zariski-open subsets $U \subseteq X$.

Sheaf Overview: A *sheaf of abelian groups* on a topological space X is, first of all, a contravariant functor:

$$\mathcal{F} : \{\text{open subsets of } X\} \to \{\text{abelian groups}\}$$

from the category of open subsets of X to the category of abelian groups, which is to say that $\mathcal{F}(U)$ is an abelian group for each open set $U \subseteq X$ and to each inclusion $U \subseteq V$ is associated a "restriction" group homomorphism:

$$\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$$

with the following properties:

(i) $\rho_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ and

(ii) $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U} : \mathcal{F}(W) \to \mathcal{F}(U)$ whenever $U \subseteq V \subseteq W$.

Finally, there are two additional axioms that a sheaf is required to satisfy. For all open covers $U = \cup_{\lambda \in \Lambda} U_\lambda$ one requires:

(a) If $s \in \mathcal{F}(U)$ satisfies $\rho_{U,U_\lambda}(s) = 0$ for all $\lambda \in \Lambda$, then $s = 0$.

(b) Given $s_\lambda \in \mathcal{F}(U_\lambda)$ with $\rho_{U_\lambda, U_\lambda \cap U_\mu}(s_\lambda) = \rho_{U_\mu, U_\lambda \cap U_\mu}(s_\mu)$ for all λ, μ, then there is an $s \in \mathcal{F}(U)$ (unique by (a)) such that $\rho_{U,U_\lambda}(s) = s_\lambda$ for all λ.

And if the functor is to the category of commutative rings (with 1) then $\mathcal{F}$ is a *sheaf of commutative rings (with 1)*.

Role in Geometry: A differentiable manifold M admits an open cover $M = \cup U_\lambda$ such that the U_λ are homeomorphic (via h_λ) to open sets in $\mathbf{R}^n$ and such that the "glueing functions" $h_\mu \circ h_\lambda^{-1} : h_\lambda(U_\lambda \cap U_\mu) \xrightarrow{\sim} h_\mu(U_\lambda \cap U_\mu)$ are not just homeomorphisms but diffeomorphisms. The notion of a function $f : U \to \mathbf{R}$ being $\mathcal{C}^\infty$ is then well-defined (checked on subsets of $\mathbf{R}^n$) and:

$$\mathcal{C}^\infty_M(U) := \{\mathcal{C}^\infty \text{ functions } f : U \to \mathbf{R}\}$$

defines a sheaf of commutative rings, with $\rho_{V,U}$ the ordinary restriction of functions. Properties (a) and (b) are built into the definition of a function!

From this point of view, a mapping of differentiable manifolds $\Phi : M \to N$ is itself differentiable if it is continuous and if, for all open subsets $U \subset N$:

$$\Phi^* : \mathcal{C}^\infty_N(U) \to \mathcal{C}^\infty_M(\Phi^{-1}(U));\ \ \Phi^*(f) = f \circ \Phi$$

that is, if $\mathcal{C}^\infty$ functions on (open sets of) N pull back to $\mathcal{C}^\infty$ functions on M.

Back to Regular Functions: The sheaf of regular functions $\mathcal{O}_X$ defined above on an affine variety is a sheaf of commutative rings. In this case, ρ_{UV} is the identity map, when viewed as a map of rational functions $\phi \in \mathbf{C}(X)$. Again, property (b) is automatic, but property (a) merits a check, just to make sure that zero (as a function) is the same as zero (as a rational function).

Proposition 3.4: If $0 = \phi \in \mathcal{O}_X(U)$ as a function on **any** (nonzero) open set $U \subset X$, then $0 = \phi \in \mathbf{C}(X)$ as a rational function .

Proof: If $\phi = \frac{\overline{f}}{\overline{g}}$ where $\overline{g}$ is regular at a point of U, then by assumption:

$$\overline{f}(a_1, ..., a_n) = 0 \text{ for all } (a_1, ..., a_n) \in U - V(\overline{g})$$

But all nonempty open sets are dense in X (as in Proposition 3.2(b)) so it follows that $\overline{f}(a_1, ..., a_n) = 0$ for all points of X (the zero locus is closed!). But then $\overline{f} = 0 \in \mathbf{C}[X]$, so $\phi = \frac{0}{\overline{g}} \in \mathbf{C}(X)$ is indeed zero.

In the spirit of differentiable maps of differentiable manifolds, we define:

Definition: A map $\Phi : X \to Y$ of affine varieties is a *regular map* if:

(a) Φ is continuous, as a map of (Zariski) topological spaces

(b) $\Phi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\Phi^{-1}(U))$. That is, when $\phi \in \mathbf{C}(Y)$ is regular at all points of U, there is a (unique by Proposition 3.4) $\psi \in \mathbf{C}(X)$, regular at the points of $\Phi^{-1}(U)$, such that $\psi = \phi \circ \Phi : \Phi^{-1}(U) \to \mathbf{C}$. Or, to put it succinctly, Φ pulls back regular functions to regular functions.

Proposition 3.5: For affine varieties $X \subset \mathbf{C}^n$ and $Y \subset \mathbf{C}^m$, a mapping $\Phi : X \to Y$ is a regular map if and only if there are polynomials:

$$f_1, ..., f_m \in \mathbf{C}[x_1, ..., x_n] \text{ such that } \Phi = (f_1, ..., f_m) : X \to Y$$

and so there is a natural bijection:

$$\{\text{regular maps } \Phi : X \to Y\} \leftrightarrow$$

$$\{\mathbf{C}\text{-algebra homomorphisms } \Phi^* : \mathbf{C}[Y] \to \mathbf{C}[X]\}$$

Proof: Let $y_1, ..., y_m$ be the coordinates on $\mathbf{C}^m$. Then $\overline{y}_1, ..., \overline{y}_m \in \mathbf{C}[Y]$ are regular functions, i.e. elements of $\mathcal{O}_Y(Y)$. By the definition, if $\Phi : X \to Y$ is a regular map, then $\Phi^*(\overline{y}_1), ..., \Phi^*(\overline{y}_m) \in \mathcal{O}_X(X)$. But by Proposition 3.3, $\mathcal{O}_X(X) = \mathbf{C}[X]$, so $\Phi^*(\overline{y}_1) = \overline{f}_1, ..., \Phi^*(\overline{y}_m) = \overline{f}_m$ for some regular functions $\overline{f}_1, ..., \overline{f}_m \in \mathbf{C}[X]$, and then indeed:

$$\Phi(a_1, ..., a_n) = (f_1(a_1, ..., a_n), ..., f_m(a_1, ..., a_n))$$

for any choice of representatives $f_1, .., f_m \in \mathbf{C}[x_1, ..., x_n]$.

Conversely, $\mathbf{C}$-algebra homomorphisms $\Phi^* : \mathbf{C}[Y] \to \mathbf{C}[X]$ are set by choosing the regular functions $\overline{f}_1 = \Phi^*(\overline{y}_1), .., \overline{f}_m = \Phi^*(\overline{y}_m)$ (arbitrarily!). Thus the maps $\Phi = (\overline{f}_1, ..., \overline{f}_m) : X \to Y$ are in a natural bijection with the $\mathbf{C}$-algebra homomorphisms $\Phi^* : \mathbf{C}[Y] \to \mathbf{C}[X]$, since any such Φ is clearly a regular map.

Remark: If we think of the affine varieties (with sheaves of regular functions) as a category with regular maps as the morphisms, then Proposition 3.5 shows that this is the same as the category of domains $\mathbf{C}[x_1, ..., x_n]/P$ with $\mathbf{C}$-algebra homomorphisms (going in the opposite direction!) via $X \mapsto \mathbf{C}[X]$ and $\Phi \mapsto \Phi^*$. That is, any property of affine varieties translates into an equivalent property of such domains and vice versa.

So why bother with affine varieties at all? The reason is geometric. Since affine varieties, unlike domains, are topological spaces with sheaves on them, we can ask the following questions of an *arbitrary* topological space X with a sheaf $\mathcal{O}_X$ of commutative rings on it.

(1) Is X with the sheaf $\mathcal{O}_X$ isomorphic to an affine variety? Or,

(2) Does there exist an open cover $X = \cup U_\lambda$ such that the (topological) spaces U_λ with induced sheaves $\mathcal{O}_{U_\lambda}$ are isomorphic to affine varieties?

Example (Quasi-Affine Varieties): Let $W \subset X$ be an open subset of an affine variety, with induced sheaf $\mathcal{O}_W$ of regular functions defined by:

$$\mathcal{O}_W(U) := \mathcal{O}_X(U)$$

for all open subsets $U \subset W$. This topological space W with its sheaf $\mathcal{O}_W$ is called a *quasi-affine* variety. A *regular map* $\Phi : W \to Y$ to an affine variety (or another quasi-affine variety) is defined by requiring Φ to be (as always!) (a) continuous, and (b) pull back regular functions to regular functions.

Proposition 3.6: Each **basic** open quasi-affine subset $W = X - V(\overline{f}) \subset X$ of an affine variety X is isomorphic to the affine "hyperbola variety over W":

$$Y := V(\langle P, 1 - x_{n+1}f\rangle) \subset \mathbf{C}^{n+1}$$

where $X = V(P) \subset \mathbf{C}^n$ and $f \in \mathbf{C}[x_1, ..., x_n]$ is any representative of $\overline{f}$.

Proof: We need a bijection $\Phi : W \to Y$ so that Φ and Φ^{-1} are regular. Here it is:

$$\Phi(a_1, ..., a_n) = (a_1, ..., a_n, \frac{1}{f(a_1, ..., a_n)})$$

with inverse $\Phi^{-1}(b_1, ..., b_{n+1}) = (b_1, ..., b_n)$. The inverse Φ^{-1} is regular as a map from Y to X by Proposition 3.5, since it is given by the polynomials $x_1, ..., x_n$, and it follows that Φ^{-1} is also regular as a map to W.

As for Φ itself, if $U = Y - V(\overline{g})$ is a basic open set, then

$$\Phi^{-1}(U) = \{(a_1, ..., a_n) \in W \mid g(a_1, ..., a_n, \frac{1}{f(a_1, ..., a_n)}) \neq 0\}$$

is open since $g(x_1, ..., x_n, f^{-1}(x_1, ..., x_n)) = \frac{h(x_1,...,x_n)}{f^N}$ for some $h \in \mathbf{C}[x_1, ..., x_n]$ and then $\Phi^{-1}(U) = W - V(\overline{h})$. Since these sets generate the topology on Y, this is enough to see that Φ is continuous.

And if $\phi = \frac{\overline{k}}{\overline{g}} \in \mathbf{C}(Y)$ is regular at $(a_1, ..., a_n, f^{-1}(a_1, ..., a_n)) \in U$ then

$$\Phi^*(\phi) = \frac{k(x_1, ..., x_n, f^{-1})}{g(x_1, ..., x_n, f^{-1})} = \frac{l(x_1, ..., x_n)}{f^M h(x_1, ..., x_n)}$$

for some M and l, and this is a rational function on X which is regular at the point $(a_1, ..., a_n) = \Phi^{-1}(a_1, ..., a_n, f^{-1}(a_1, ..., a_n))$. So (b) is satisfied.

Corollary 3.7: Every quasi-affine variety has an open cover by quasi-affine varieties that are isomorphic to affine varieties.

Proof: If Y is quasi-affine, then $Y \subset X$ is open in some affine variety, and then Y is covered by basic open sets in X which are isomorphic to affine "hyperbola" varieties by Proposition 3.6.

Exercises 3.

1. (a) Show that the Zariski topology on $\mathbf{C}^2$ and the product of the Zariski topologies on $\mathbf{C}^2 = \mathbf{C}^1 \times \mathbf{C}^1$ are not the same.

(b) Prove that each non-empty open subset in an affine variety is dense.

(c) If $X \subseteq \mathbf{C}^n$ is an affine variety with $\mathbf{C}[X] = \mathbf{C}[x_1, ..., x_n]/P$, and if $I \subseteq \mathbf{C}[X]$ and $P \subseteq J \subseteq \mathbf{C}[x_1, ..., x_n]$ are ideals such that $J/P = I$, prove that $\sqrt{J}/P = \sqrt{I}$, and conclude that, as claimed in the text:

$$\{\text{closed subsets of } X\} \leftrightarrow \{\text{radical ideals in } \mathbf{C}[X]\}$$

2. Prove that if X an affine variety, then the contravariant functor:

$$\mathbf{Z}(U) = \mathbf{Z}; \quad \text{with identity restrictions } \rho_{VU} = \mathrm{id}_{\mathbf{Z}}$$

defines a sheaf of abelian groups, the "constant sheaf of integers" on X.

On the other hand, if X is a topological space which is not irreducible (i.e. if there are proper closed subsets $Z_1, Z_2 \subset X$ such that $Z_1 \cup Z_2 = X$) prove that this is not a sheaf. Fix the definition in that case to make a sheaf!

3. If $Z \subset X$ is a closed subset of an affine (or quasi-affine) variety X, define the *sheaf of ideals* of Z by:

$$I_Z(U) := \{\text{regular functions on } U \text{ that vanish along } Z \cap U\} \subseteq \mathcal{O}_X(U)$$

Prove that this is a sheaf of abelian groups.

4. (a) If $U = X - V(\overline{g})$ is a basic open subset of an affine variety, prove that:

$$\mathcal{O}_X(U) = \mathbf{C}[X]_{\overline{g}} := \left\{ \frac{\overline{f}}{\overline{g}^m} \mid \overline{f} \in \mathbf{C}[X], m \geq 0 \right\} \subset \mathbf{C}(X)$$

(b) Find $\mathcal{O}_{\mathbf{C}^n}(U)$ when U is the complement of the origin in $\mathbf{C}^n$.

(c) Prove that when $n > 1$, the quasi-affine varieties $U = \mathbf{C}^n - 0$ are *not* isomorphic to any affine varieties.

5. (a) A subset $Y \subseteq X$ is *locally closed* if Y is the intersection $U \cap Z$ of an open and closed set. Prove that an intersection of two locally closed sets is locally closed, but that the union $(\mathbf{C}^2 - \{y-\text{axis}\}) \cup \{0\}$ is not locally closed.

(b) If X is a quasi-affine variety and $Y \subset X$ is an irreducible locally closed subset, give Y the induced topology and define, for open subsets $U \subset Y$:

$$\mathcal{O}_Y(U) := \{\overline{\phi} : U \to \mathbf{C} \mid \exists V \subset X \text{ with } V \cap Y = U, \text{ an open cover } V = \cup V_i$$

$$\text{and } \phi_i \in \mathcal{O}_X(V_i)) \text{ such that } \overline{\phi}|_{V_i \cap Y} = \phi_i|_{V_i \cap Y}\}$$

Prove first that $\mathcal{O}_Y$ is a sheaf, and then show that it is the same sheaf one obtains by regarding $Y \subseteq \overline{Y} \subset \overline{X} \subseteq \mathbf{C}^n$ as a quasi-affine variety.

(c) If $\Phi : X \to X'$ is a regular map of quasi-affine varieties, prove that the induced maps $\Phi|_Y : Y \to X'$ are regular maps, for all irreducible locally closed subsets $Y \subseteq X$ and conclude that if Φ is an isomorphism, then Φ induces isomorphisms $\Phi|_Y : Y \to Y' = \Phi(Y)$ of all such quasi-affine varieties.

6. Consider the *cuspidal cubic curve*: $X = V(y^2 - x^3) \subset \mathbf{C}^2$.

(a) Prove that the mapping: $\Phi : \mathbf{C} \to X; \quad t \mapsto (t^2, t^3)$ is a regular map and a homeomorphism of Zariski topological spaces.

(b) Prove that the inverse mapping $\Phi^{-1} : X \to \mathbf{C}$ is **not** a regular map.

7. (a) Prove that $\mathbf{C}$ is not isomorphic (as an affine variety) to $\mathbf{C}^* = \mathbf{C} - \{0\}$.

(b) More generally, prove that $\mathbf{C} - \{p_1, ..., p_m\}$ and $\mathbf{C} - \{q_1, ..., q_n\}$ are never isomorphic if $m \neq n$.

(c) Prove that $\mathbf{C} - \{p_1, p_2\}$ and $\mathbf{C} - \{q_1, q_2\}$ are always isomorphic varieties. How many different isomorphisms are there between them?

(d) What about $\mathbf{C} - \{p_1, ..., p_n\}$ and $\mathbf{C} - \{q_1, ..., q_n\}$ in general?