Complex Algebraic Geometry: Varieties

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3. Projective Varieties. To first approximation, a projective variety is the locus of zeroes of a system of *homogeneous* polynomials:

$$F_1, \ldots, F_m \in \mathbb{C}[x_1, \ldots, x_{n+1}]$$

in projective n-space. More precisely, a projective variety is an abstract variety that is isomorphic to a variety determined by a homogeneous prime ideal in $\mathbb{C}[x_1, \ldots, x_{n+1}]$. Projective varieties are *proper*, which is the analogue of "compact" in the category of abstract varieties.

Projective n-space $\mathbb{P}^n$ is the set of lines through the origin in $\mathbb{C}^{n+1}$.

The *homogeneous "coordinate"* of a point in $\mathbb{P}^n$ (= line in $\mathbb{C}^{n+1}$) is:

$$(x_1 : \cdots : x_{n+1}) \quad \text{(not all zero)}$$

and it is well-defined modulo:

$$(x_1 : \cdots : x_{n+1}) = (\lambda x_1 : \cdots : \lambda x_{n+1}) \text{ for } \lambda \in \mathbb{C}^*$$

Projective n-space is an overlapping union:

$$\mathbb{P}^n = \bigcup_{i=1}^{n+1} U_i; \quad U_i = \{(x_1 : \cdots : x_m : \cdots x_{n+1}) \mid x_m \neq 0\} = \mathbb{C}^n$$

and a disjoint union:

$$\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \mathbb{C}^{n-2} \cup \cdots \cup \mathbb{C}^0$$

where $\mathbb{C}^{n-m} = \{(x_1 : \cdots : x_{n+1}) \mid x_1 = \cdots = x_m = 0, x_{m+1} \neq 0\}$

The polynomial ring is graded by degree:

$$\mathbb{C}[x_1, \ldots, x_{n+1}] = \bigoplus_{d=0}^{\infty} \mathbb{C}[x_1, \ldots, x_{n+1}]_d$$

and the nonzero polynomials $F \in \mathbb{C}[x_1, \ldots, x_{n+1}]_d$ are the *homogeneous polynomials of degree d*. The value of a homogeneous polynomial F of degree d at a point $x \in \mathbb{P}^n$ is not well-defined, since:

$$F(\lambda x_1, \ldots, \lambda x_{n+1}) = \lambda^d F(x_1, \ldots, x_{n+1})$$

However, the *locus of zeroes* of F is well-defined, hence F determines a *projective hypersurface* if $d > 0$:

$$V(F) = \{(x_1 : \cdots : x_{n+1}) \in \mathbb{P}^n \mid F(x_1 : \cdots : x_{n+1}) = 0\} \subset \mathbb{P}^n,$$

Definition: An ideal $I \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$ is *homogeneous* if:

$$I = \bigoplus I_d; \text{ where } I_d = I \cap \mathbb{C}[x_1, \ldots, x_{n+1}]_d$$

equivalently, I has (finitely many) homogeneous generators.

Corollary 3.1: For a homogeneous ideal $I \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$,

$$V(I) = \{(x_1 : \cdots : x_{n+1}) \mid F(x_1 : \cdots : x_{n+1}) = 0 \text{ for all } F \in I\} \subset \mathbb{P}^n$$

is an intersection of finitely many projective hypersurfaces.

The sets $V(I) \subset \mathbb{P}^n$ are the *algebraic subsets* of $\mathbb{P}^n$. The *irreducible* algebraic sets are defined as in the case of affine varieties, and satisfy:

$$X = V(\mathcal{P}) \subset \mathbb{P}^n$$

for a unique homogeneous *prime* ideal $\mathcal{P} \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$. Thus each irreducible algebraic set *inside* $\mathbb{P}^n$ has a *homogeneous coordinate ring:*

$$R(X) := \mathbb{C}[x_1, \ldots, x_n]/\mathcal{P} = \bigoplus \mathbb{C}[x_1, \ldots, x_n]_d/\mathcal{P}_d$$

Warning: Unlike the coordinate rings of isomorphic affine varieties, homogeneous rings of isomorphic projective varieties will **not** usually be isomorphic graded rings. Even more fundamentally, a non-constant element of the homogeneous ring of $X \subset \mathbb{P}^n$ is **not** a function.

(In fact, homogeneous rings are made up of sections of line bundles.)

Note: There is one homogeneous maximal ideal, namely:

$$\langle x_1, \ldots, x_{n+1} \rangle \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$$

This is usually called the *irrelevant* homogeneous ideal, and it contains all other homogeneous ideals.

Definition: A homogeneous ideal $m \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$ is *homaximal* if it is maximal among all homogeneous ideals other than $\langle x_1, \ldots, x_{n+1} \rangle$.

Exercises 3.1:

(a) Given a homogeneous $I \subset \langle x_1, \ldots, x_{n+1} \rangle$, there is a bijection:

$$\{\text{homaximal ideals } m_x \text{ containing } I\} \leftrightarrow \{\text{points } x \in V(I) \subset \mathbb{P}^n\}$$

(b) There is a (Zariski) topology on $X = V(\mathcal{P})$ generated by:

$$U_F := X - V(F) \subset \mathbb{P}^n$$

for $F \in R(X)_d$ consisting of open sets of the form $U_I := X - V(I)$ for homogeneous ideals $I \subset R(X)$.

Definition: The field of rational functions on $X = V(\mathcal{P}) \subset \mathbb{P}^n$ is:

$$\mathbb{C}(X) := \left\{ \frac{F}{G} \mid F, G \in R(X)_d \text{ for some } d, \text{ and } G \neq 0 \right\} / \sim$$

Good News: Rational functions are $\mathbb{C}$-valued functions on some U.

Bad News/Exercise 3.2: The only rational functions defined everywhere on X are the constant functions.

Proposition 3.2: The sheaf $\mathcal{O}_X$ of $\mathbb{C}$-valued functions on (the open sets of) an irreducible algebraic set $X = V(\mathcal{P}) \subset \mathbb{P}^n$ defined by:

$$\mathcal{O}_x := \left\{ \frac{F}{G} \mid G(x) \neq 0 \right\} \subset \mathbb{C}(X) \text{ and } \mathcal{O}_X(U) := \bigcap_{x \in U} \mathcal{O}_x \subset \mathbb{C}(X)$$

gives $(X, \mathcal{O}_X)$ the structure of a prevariety.

Proof: We need to prove $(X, \mathcal{O}_X)$ is locally affine.

For each $i = 1, \dots, n+1$, either:

(a) $x_i \in \mathcal{P}$, in which case $X \cap U_i = \emptyset$, or else

(b) $x_i \notin \mathcal{P}$, in which case $U_{x_i} = X \cap U_i$, with $\mathcal{O}_X|_{U_{x_i}}$ is isomorphic to the affine variety corresponding to the $\mathbb{C}$-algebra:

$$\mathbb{C}[U_{x_i}] = \mathbb{C}\left[\frac{x_1}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right] / \widetilde{\mathcal{P}}, \text{ where } F(\frac{x_1}{x_i}, \cdots, \frac{x_{n+1}}{x_i}) \in \widetilde{\mathcal{P}} \Leftrightarrow F \in \mathcal{P}$$

which, incidentally, satisfies $\mathbb{C}(U_{x_i}) = \mathbb{C}(X)$. Since X is covered by these open sets, it follows that X is locally affine.

Definition: A prevariety is *projective* if it is isomorphic to one of the $X = V(\mathcal{P}) \subset \mathbb{P}^n$ with sheaf of $\mathbb{C}$-valued functions defined as above.

As for separatedness, first notice:

Proposition 3.3: $\mathbb{P}^m \times \mathbb{P}^n$ is projective, and $\mathbb{P}^n$ is separated.

Proof: The *Segre embedding* is the map (of sets):

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$

$$((x_1 : \cdots : x_{n+1}), (y_1 : \cdots : y_{m+1})) \mapsto (\cdots : x_i y_j : \dots)$$

(We will use z_{ij} as homogeneous coordinates for points of $\mathbb{P}^{(n+1)(m+1)-1}$.) The image of σ is the irreducible algebraic set:

$$X_{m,n} := V(\{z_{ij}z_{kl} - z_{il}z_{kj}\}) \subset \mathbb{P}^{(n+1)(m+1)-1}$$

and moreover, as a prevariety, $X_{m,n}$ (with sheaf of regular functions) is the product of $\mathbb{P}^n$ and $\mathbb{P}^m$. Also, if $n = m$, then:

$$\delta(\mathbb{P}^n) = X_{n,n} \cap V(\{z_{ij} - z_{ji}\})$$

is closed, so $\mathbb{P}^n$ is separated.

Exercise 3.3: (a) Carefully show that the projection $\pi_{\mathbb{P}^m} : X_{m,n} \to \mathbb{P}^m$ is a morphism of prevarieties.

(b) Extend Proposition 3.3 to describe the product of projective prevarieties $X = V(\mathcal{P}) \subset \mathbb{P}^n$ and $Y = V(\mathcal{Q}) \subset \mathbb{P}^m$ as an irreducible, closed subset of $X_{m,n}$, hence it is a projective prevariety, and then conclude that all projective prevarieties are varieties.

Definition: An open subset $U \subset X$ of a projective variety, together with the induced sheaf $\mathcal{O}_X|_U$ of $\mathbb{C}$-valued functions, or more generally, any variety isomorphic to such a pair $(U, \mathcal{O}_X|_U)$, is *quasi-projective.*

Proposition 3.4: A quasi-affine variety is also quasi-projective.

Proof: It suffices to show that each affine variety is quasi-projective. To this end, suppose $(Y, \mathcal{O}_Y)$ is isomorphic to the affine variety obtained from $X = V(\mathcal{P}) \subset \mathbb{C}^n$. Then we may identify $\mathbb{C}^n$ with $U_{n+1} \subset \mathbb{P}^n$ and take the (Zariski) closure $\overline{X}$ of $X \subset \mathbb{P}^n$. Then

$$\overline{X} = V(\mathcal{P}^h), \text{ where } \mathcal{P}^h = \langle f^h \mid f \in \mathcal{P}\rangle$$

and $f^h(x_1, \ldots, x_{n+1}) := x_{n+1}^d f(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}})$ whenever $d = \deg(f)$. Check that $\mathcal{P}^h$ defined this way is a prime ideal, and that $\overline{X} \cap \mathbb{C}^n = X$.

Exercise 3.4: Find generators $\langle f_1, \cdots f_m\rangle = \mathcal{P}$ of a prime ideal with the property that the f_i^h do **not** generate $\mathcal{P}^h$.

Definition: In a category whose objects are topological spaces, whose morphisms are continuous, and in which products exist, a separated object X is *proper* if "projecting from X is universally closed," i.e.

$$\pi_Y : X \times Y \to Y$$

maps closed sets $Z \subset X \times Y$ to closed sets $\pi_Y(Z) \subset Y$, for all Y.

Exercise 3.5: In the category of topological spaces, compact implies proper, and conversely, any proper space with the property that every open cover has a *countable* subcover is also compact.

Concrete Example: Suppose $(X, \mathcal{O}_X)$ is affine and $f \in \mathbb{C}[X]$ is a non-constant function. Then the hyperbola over U_f:

$$V(xf - 1) \subset X \times \mathbb{C}$$

is closed, but its projection to $\mathbb{C}$ has image $\mathbb{C}^*$, which is not closed. Thus, the **only** proper affine variety is the one-point space.

General Example: If $U \subset X$ is an open subset of a separated object and $U \neq \overline{U}$ (e.g. any quasi-projective variety properly contained in a projective variety), then U is not proper. Indeed, the closed diagonal in $X \times X$ determines a closed set:

$$\Delta \cap (U \times X)$$

that projects to $U \subset X$.

Theorem (Grothendieck) Projective varieties are proper varieties.

To prove this, we need a result from commutative algebra:

Nakayama's Lemma (Version 1): Suppose M is a finitely generated module over a ring A and $I \subset A$ is an ideal such that:

$$IM = M$$

Then there an element $a \in I$ such that $(1+a)m = 0$ for all $m \in M$.

Proof: Let $m_1, \ldots, m_n \in M$ be generators. Then $IM = M$ gives:

$$m_i = \sum_{j=1}^{n} a_{ij} m_j \text{ for } a_{ij} \in I$$

implying that the matrix $1-(a_{ij})$ has a kernel, hence $b = \det(1-(a_{ij}))$ satisfies $bm_i = 0$ for all i, and evidently, $b = 1+a$ for some $a \in I$.

Proof: (of the theorem) It suffices to prove that for all n, m:

$$\pi_{\mathbb{C}^m} : \mathbb{P}^n \times \mathbb{C}^m \to \mathbb{C}^m \text{ is a closed map}$$

Let $\mathbb{C}[x_1, \cdots x_{n+1}] = R(\mathbb{P}^n)$ and $\mathbb{C}[y_1, \cdots, y_m] = \mathbb{C}[\mathbb{C}^m]$, and suppose

$$Z \subset \mathbb{P}^n \times \mathbb{C}^m \text{ is closed}$$

Then we need to show that $\pi_{\mathbb{C}^m}(Z) = V(J)$ for some $J \subset \mathbb{C}[y_1, \ldots, y_m]$. To this end, grade the ring $\mathbb{C}[\overline{x}, \overline{y}] := \mathbb{C}[x_1, \cdots, x_{n+1}, y_1, \cdots, y_m]$ by degree in the x-variables:

$$\mathbb{C}[\overline{x}, \overline{y}] = \bigoplus_d \mathbb{C}[x_1, \cdots, x_{n+1}]_d \otimes \mathbb{C}[y_1, \cdots, y_m]$$

and consider the subsets defined by homogeneous ideals:

$$V(I) \subset \mathbb{P}^n \times \mathbb{C}^m; \quad I = \bigoplus_d I_d \subset \bigoplus_d \mathbb{C}[\overline{x}, \overline{y}]_d$$

Then the theorem follows immediately from:

Claim 1: Every closed set $Z \subset \mathbb{P}^n \times \mathbb{C}^m$ is equal to some $V(I)$.

Claim 2: $\pi_{\mathbb{C}^m}(V(I)) = V(I_0)$, the "degree zero" part of the ideal I.

Proof of Claim 1: Cover $\mathbb{P}^n$ by the open sets $U_i = \mathbb{C}^n \times \mathbb{C}^m$ with:

$$\mathbb{C}[U_i] = \mathbb{C}[\frac{x_1}{x_i}, \ldots, \frac{x_{n+1}}{x_i}, y_1, \ldots, y_m]$$

and, given a closed set $Z \subset \mathbb{P}^n \times \mathbb{C}^m$, define a homogeneous ideal I by:

$$I_d := \left\{ F \in \mathbb{C}[\overline{x}, \overline{y}]_d \mid \frac{F}{x_i^d} \in I(Z \cap U_i) \subset \mathbb{C}[U_i] \text{ for all } i \right\}$$

It is clear that $Z \subseteq V(I)$. For the other inclusion, suppose $(a, b) \notin Z$. Then $(a, b) \in U_i$ for some i, so there is an $f \in I(Z \cap U_i)$ such that $f(a, b) \neq 0$. It follows that $x_i^d f \in \mathbb{C}[U_i]$ for some d, and then that $F := x_i^{d+1} f \in I_d$. Since $F(a, b) \neq 0$, it follows that $(a, b) \notin V(I)$.

Proof of Claim 2: Again, the inclusion $\pi_{\mathbb{C}^m}(Z) \subset V(I_0)$ is clear. For the other inclusion, suppose $b = (b_1, \ldots, b_m) \notin \pi_{\mathbb{C}^m}(Z)$, and let $m_b = \langle y_1 - b_1, \ldots, y_m - b_m \rangle$ be the corresponding maximal ideal. Then:

$$Z \cap (\mathbb{P}^n \times b) = \emptyset, \text{ so } (Z \cap U_i) \cap (U_i \times b) = \emptyset \text{ for all } i$$

which in turn implies that:

$$I(Z \cap U_i) + \mathbb{C}\left[\frac{x_1}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right] \cdot m_b = \mathbb{C}[U_i] \text{ for all } i$$

and thus for each $i = 1, \ldots, n+1$, there exist $f_i \in I(Z \cap U_i), g_{ij} \in \mathbb{C}[U_i]$ and $m_{ij} \in m_b$ such that $f_i + \sum_j g_{ij} m_{ij} = 1$. Moreover, by multiplying through by a sufficiently large power d_i of each x_i, we can arrange that:

$$F_i + \sum_j G_{ij} m_{ij} = x_i^{d_i} \quad \text{for} \quad F_i \in I_{d_i}, G_{ij} \in \mathbb{C}[\overline{x}, \overline{y}]_{d_i}$$

If we moreover take $d > \sum d_i$, then we have $I_d + \mathbb{C}[\overline{x}, \overline{y}]_d \cdot m_b = \mathbb{C}[\overline{x}, \overline{y}]_d$.

Thus the finitely generated $\mathbb{C}[y_1, \ldots, y_m]$ -modules:

$$M_d := \mathbb{C}[\overline{x}, \overline{y}]_d / I_d \text{ satisfy } m_b \cdot M_d = M_d$$

hence by Nakayama's lemma, there is an $f \in \mathbb{C}[y_1, \cdots, y_m]$ such that $f(b) \neq 0$ and $f M_d = 0$, i.e. $f \cdot \mathbb{C}[\overline{x}, \overline{y}]_d \in I_d$. But this implies $f \cdot x_i^d \in I_d$ for all i, from which it follows that $f \in I_0$, as desired.

Corollary 3.5/Exercise 3.6: Any morphism $\Phi : X \to Y$ from a projective variety X to an abstract variety Y is a closed mapping.