Complex Algebraic Geometry: Varieties

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2. Abstract Varieties. An abstract variety is a set with a (Zariski) topology and a sheaf of functions that is locally affine and separated. In order to define these terms properly, we need to define them in the context of an appropriate category.

Let X be a topological space and A be a commutative ring with 1.

Definition. A *sheaf* $\mathcal{S}$ *of* A*-valued functions* on X consists of:

(a) A commutative ring with 1, denoted $\mathcal{S}(U)$, consisting of a subring of the ring of functions $f : U \to A$ for each open $U \subset X$, such that:

(b) functions in $\mathcal{S}(V)$ restrict to functions in $\mathcal{S}(U)$ if $U \subseteq V$, and

(c) functions $f_i \in \mathcal{S}(U_i)$ that agree when restricted to intersections $U_i \cap U_j$ are restrictions of a single well-defined function $f \in \mathcal{S}(\cup U_i)$.

Examples: (a) The sheaf of "totally discontinuous" functions on X. In this sheaf, denoted A_X^{disc}, each $A_X^{\text{disc}}(U)$ consists of all $f : U \to A$.

(b) At the other extreme, the constant functions do not form a sheaf. But there is a sheaf, denoted A_X, consisting of functions that are *locally constant*. In this sheaf, $A_X(U)$ consists of all the functions $f : U \to A$ that are constant *on each connected component* of U.

(c) If A has a topology (e.g. $\mathbb{R}$ or $\mathbb{C}$ with the Euclidean topology), then the functions $f : U \to A$ that are *continuous* form a sheaf, often denoted simply by $\mathcal{C}_X$. This generalizes examples (a) and (b) (Why?)

(d) If X is a differentiable manifold, then the rings of "infinitely differentiable" functions $f : U \to \mathbb{R}$ form a sheaf, denoted by $\mathcal{C}_X^\infty$.

(e) The regular functions on an affine variety X with the Zariski topology form a sheaf of $\mathbb{C}$-valued functions, denoted $\mathcal{O}_X$, which has the unusual property that $U \subseteq V$ implies $\mathcal{O}_X(V) \subseteq \mathcal{O}_X(U)$. In other words, the restriction of such functions is an *injective* map for this sheaf. This is not true of any of the other sheaves discussed here (Why?)

(f) Suppose $V \subset X$ is an open subset and $\mathcal{S}$ is a sheaf of A-valued functions on X. Then the induced topology on V, together with the "restricted sheaf" $\mathcal{S}|_V$ defined by: $\mathcal{S}|_V(U) := \mathcal{S}(U)$ for all $U \subseteq V$ is a sheaf of A-valued functions on V.

Definition: A *morphism* between pairs $(X, \mathcal{S}_X)$ and $(Y, \mathcal{S}_Y)$ consisting of a topological space with a sheaf of A-valued functions is:

(a) A continuous map $F : X \to Y$ with the property that:

(b) the pull-back on functions defined by $F^*(f) := f \circ F$ maps each:

$$F^* : \mathcal{S}_Y(U) \to \mathcal{S}_X(F^{-1}(U))$$

This gives the collection of pairs $(X, \mathcal{S}_X)$ the structure of a *category*.

Examples: (a) For any X, the identity map defines a morphism:

$$\text{id} : (X, A_X^{\text{disc}}) \to (X, A_X)$$

but not in the opposite direction (unless X is discrete)! Similarly,

(b) The identity map on a differentiable manifold X defines:

$$\text{id} : (X, \mathcal{C}_X) \to (X, \mathcal{C}_X^\infty)$$

but not in the opposite direction.

(c) If $\iota : V \subset X$ is an open set together with the restricted sheaf $\mathcal{S}|_V$ (for any sheaf $\mathcal{S}$ of A-valued functions), then the inclusion map

$$\iota : (V, \mathcal{S}|_V) \to (X, \mathcal{S})$$

is a morphism.

(d) A continuous mapping of differentiable manifolds $F : X \to Y$ is (infinitely) differentiable if and only if it defines a morphism:

$$F : (X, \mathcal{C}_X^\infty) \to (Y, \mathcal{C}_Y^\infty)$$

i.e. it pulls back (locally) infinitely differentiable functions on Y to (locally) infinitely differentiable functions on X.

(e) Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be irreducible algebraic sets, with Zariski topologies and coordinate rings $\mathbb{C}[X]$ and $\mathbb{C}[Y]$. A morphism $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in the category of topological spaces with sheaves of $\mathbb{C}$-valued functions determines a $\mathbb{C}$-algebra homomorphism $F^* : \mathcal{O}_Y(Y) = \mathbb{C}[Y] \to \mathbb{C}[X] = \mathcal{O}_X(X)$, **and conversely:**

Proposition 2.1: Each $\mathbb{C}$-algebra homomorphism $\Phi : \mathbb{C}[Y] \to \mathbb{C}[X]$ comes from a *uniquely determined* morphism:

$$F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

in the sense that $\Phi = F^* : \mathbb{C}[Y] \to \mathbb{C}[X]$.

Proof: Recall that the points of X are in a natural bijection with the maximal ideals of $\mathbb{C}[X]$. Thus:

$$F(x) = y \Leftrightarrow \Phi^{-1}(m_x) = m_y$$

is well-defined, and it is the *only possible* map for which $F^* = \Phi$ on (globally defined) regular functions. Moreover,

- $F^{-1}(U_g) = U_{\Phi(g)}$ for all $g \in \mathbb{C}[Y]$, so F is continuous, and
- $F^*(f/g^n) = \Phi(f)/\Phi(g)^n$ shows that $F^* : \mathcal{O}_Y(U_g) \to \mathcal{O}_X(U_{\Phi(g)})$

from which it follows that F is a morphism, as desired.

Corollary 2.2: Irreducible algebraic sets, with rational functions:

$$(X, \mathcal{O}_X) \text{ and } (Y, \mathcal{O}_Y)$$

are isomorphic in the category of topological spaces with sheaves of $\mathbb{C}$-valued functions if and only if $\mathbb{C}[X] \cong \mathbb{C}[Y]$ as $\mathbb{C}$-algebras.

Refined Definition (of an affine variety): A *complex affine variety* is a topological space X with sheaf $\mathcal{S}_X$ of $\mathbb{C}$-valued functions that is isomorphic to some $(Y, \mathcal{O}_Y)$, where Y is an irreducible algebraic set in some $\mathbb{C}^m$ and $\mathcal{O}_Y$ is its sheaf of regular functions.

Important Example: If $X = V(\mathcal{P}) \subset \mathbb{C}^n$ and $f \in \mathbb{C}[x_1, \ldots, x_n]$, then $(U_f, \mathcal{O}_X|_{U_f})$ (for $U_f \subset X$) is an affine variety. It is isomorphic to the "affine hyperbola over U_f," namely

$$Y := V(\langle \mathcal{P}, 1 - f x_{n+1} \rangle) \subset \mathbb{C}^{n+1}$$

which is an irreducible algebraic set with coordinate ring:

$$\mathbb{C}[Y] \cong \mathbb{C}[X][x_{n+1}]/\langle 1 - f x_{n+1} \rangle \cong \mathbb{C}[X][f^{-1}] \subset \mathbb{C}(X)$$

Definition: A pair $(X, \mathcal{S}_X)$ consisting of a topological space with sheaf of $\mathbb{C}$-algebras is a *prevariety* if X is connected and covered by open sets:

$$X = \cup_{i=1}^n U_i$$

with the property that each of the pairs $(U_i, \mathcal{S}_X|_{U_i})$ is an affine variety.

Corollary (of the important example) 2.3: Every (Zariski) open subset $U \subset X$ (with sheaf $\mathcal{O}_X|_U$) of an affine variety is a prevariety.

Exercise 2.1: The open subset $U = \mathbb{C}^2 - \{(0,0)\} \subset \mathbb{C}^2$ with sheaf $\mathcal{O}_U = \mathcal{O}_{\mathbb{C}^2}|_U$ is a prevariety but *not* an affine variety.

Gluing: Let $(X, \mathcal{S}_X)$ and $(Y, \mathcal{S}_Y)$ be topological spaces with sheaves of A-valued functions that have isomorphic open subsets, specifically $U \subset X$ and $V \subset Y$ with an isomorphism $F : (U, \mathcal{S}_X|_U) \xrightarrow{\sim} (V, \mathcal{S}_Y|_V)$ Then we may "glue X and Y along F" to obtain $(Z, \mathcal{S}_Z)$ defined by:

- As a set,

$$Z = (X \coprod Y)/\sim \text{ where } x \sim F(x) \text{ for each } x \in U$$

with natural inclusion maps $\iota_X : X \subset Z$ and $\iota_Y : Y \subset Z$.

- $W \subset Z$ is open if and only if both $W \cap X$ and $W \cap Y$ are open.
- $\mathcal{S}_Z(W)$ is the set of functions $f : W \to A$ satisfying:

$$f|_{W\cap X} \in \mathcal{S}_X(W \cap X),\ f|_{W\cap Y} \in \mathcal{S}_Y(W \cap Y)$$

and $F^*(f|_{W\cap V}) = f|_{W\cap U}$.

Conversely: Given $(X, \mathcal{S}_X)$ and open sets $U \subset X$ and $V \subset X$, then $(X, \mathcal{S}_X)$ is isomorphic to the topological space with sheaf of A-valued functions obtained by gluing $(U, \mathcal{S}_X|_U)$ to $(V, \mathcal{S}_X|_V)$ along the canonical isomorphism $F : U \cap V \cong V \cap U$.

Corollary 2.3: A prevariety is obtained by gluing an affine variety to another affine variety (or prevariety) along non-empty open subsets.

Two Very Different Examples: One can glue

$$(\mathbb{C}, \mathcal{O}_{\mathbb{C}}) \text{ to } (\mathbb{C}, \mathcal{O}_{\mathbb{C}})$$

along $\mathbb{C}^* = \mathbb{C} - \{0\}$ in two different ways:

(i) Gluing along the identity isomorphism id : $\mathbb{C}^* \to \mathbb{C}^*$ produces "the affine line with doubled origin." In the framework of manifolds, this is a simple example of a non-Haudorff "fake" manifold. Unfortunately, the Zariski topology on a variety is essentially never Haudorff (since *all* pairs of nonempty open subsets of an affine variety intersect). Thus, we will have to find another way to eliminate it.

However, there is another interesting automorphism of $\mathbb{C}^*$. Since:

$$(\mathbb{C}^*, \mathcal{O}_{\mathbb{C}}|_{\mathbb{C}^*}) \cong (X, \mathcal{O}_X)$$

where $\mathbb{C}[X] \cong \mathbb{C}[x, x^{-1}]$ by the affine hyperbola construction, it follows that the $\mathbb{C}$-algebra automorphism:

$$\mathbb{C}[X] \to \mathbb{C}[X];\ x \mapsto x^{-1}$$

is associated to an automorphism F of $(\mathbb{C}^*, \mathcal{O}_{\mathbb{C}^*})$.

(ii) Gluing along $x \mapsto x^{-1}$ produces the projective line.

(We will explore this in detail later.)

Definition: A *product* of objects X, Y of a category $\mathcal{C}$ is an object, which we will denote by $X \times Y$, together with "projection" morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ with the following:

Universal property: To each triple $(Z; F, G)$ consisting of an object Z, with morphisms $F : Z \to X$ and $G : Z \to Y$, there is a unique:

$$Z \to X \times Y$$

that commutes with the morphisms to X and Y.

The product is unique (if it exists) up to a uniquely determined isomorphism. Thus we can get away the leading notation "$X \times Y$."

Example: The Cartesian product is a product in the category of sets. In the category of topological spaces, the Cartesian product, together with the *product topology*, is a product.

Lookup: The tensor product $\mathbb{C}[X] \otimes_{\mathbf{C}} \mathbb{C}[Y]$ of integral domains that are finitely generated as $\mathbb{C}$-algebras is also an integral domain, and also finitely generated as a $\mathbb{C}$-algebra.

Corollary 2.4: If X and Y are affine varieties, then

$$\mathbb{C}[X[\otimes_{\mathbb{C}}\mathbb{C}[Y] \cong \mathbb{C}[X \times Y]$$

for an affine variety $X \times Y$, which, together with morphisms π_X and π_Y associated to the inclusions $\mathbb{C}[X] \subset \mathbb{C}[X \times Y]$ and $\mathbb{C}[Y] \subset \mathbb{C}[X \times Y]$, respectively, *is the product* of X and Y in the category of affine varieties. This is due to the analogous universal property of the tensor product.

Warning: Note that $\mathbb{C}^m \times \mathbb{C}^n \cong \mathbb{C}^{m+n}$, but that, as we've already noted in an earlier exercise, the Zariski topology on this product is **not** in general equal to the product topology. This does not contradict the Example above describing the products of topological spaces(why?).

Definition: In a category whose objects are topological spaces, whose morphisms are continuous, and in which products exist, an object X is *separated* if the image of the canonical *diagonal map*

$$\delta : X \to X \times X \text{ is closed.}$$

Exercise 2.2: (a) In the category of topological spaces, prove that X is *Hausdorff* if and only if it is separated.

(b) Prove that as a set, the product $X \times Y$ of affine varieties is the Cartesian product of X and Y (this is not, however, true of schemes).

Proposition 2.5: Affine varieties are separated.

Proof: Let X be an affine variety. Via an isomorphism we may assume $X = V(\mathcal{P}) \subset \mathbb{C}^n$ with coordinates $x_1, \ldots, x_n$. Then $X \times X \subset \mathbb{C}^{2n}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$, and $\delta(X) \subset X \times X$ is the **closed** subset defined by the equations $\{x_i - y_i = 0 \mid i = 1, \ldots, n\}$.

More Involved Exercises 2.3:

(a) Prove that products exist in the category of prevarieties.

(b) A *quasi-affine* variety is, by definition, a pair $(U, \mathcal{O}_X|_U)$, where $U \subset X$ is a (non-empty) open subset of an affine variety X. Prove that quasi-affine varieties are separated.

(c) Prove that the affine line with the doubled origin is not separated.

(d) Prove that the projective line is separated.

Definition: An *abstract variety* is a separated prevariety.