Complex Algebraic Geometry: Varieties

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1. Affine Varieties. To first approximation, an affine variety is the locus of zeroes (in $\mathbb{C}^n$) of a system of polynomials:

$$f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$$

Systems of polynomials are better expressed in terms of the ideals that they generate, and two theorems by Hilbert on ideals are the starting point for an "intrinsic" treatment of affine varieties.

Theorem (Hilbert Basis): Every ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ can be generated by finitely many polynomials: $I = \langle f_1, \dots, f_m \rangle$.

Definition: Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a nonconstant polynomial. Then:

$$V(f) := \{(a_1, \dots, a_n) \in \mathbb{C}[x_1, \dots, x_n] \mid f(a_1, \dots, a_n) = 0\}$$

is the *hypersurface* in $\mathbb{C}^n$ defined as the zero locus of the polynomial f.

Definition: Let $I \subset \mathbb{C}[x_1, \dots, x_n]$ be an ideal. Then:

$$V(I) := \{(a_1, \dots, a_n) \in \mathbb{C}[x_1, \dots, x_n] \mid f(a_1, \dots, a_n) = 0 \text{ for all } f \in I\}$$

is the *algebraic subset* of $\mathbb{C}^n$ determined by the polynomials in I.

Corollary 1.1: An algebraic subset of $\mathbb{C}^n$ is either $\mathbb{C}^n$ itself, or else it is the intersection of finitely many hypersurfaces.

Theorem (Hilbert Nullstellensatz): The natural map:

$$m : \mathbb{C}^n \to \{\text{maximal ideals in } \mathbb{C}[x_1, \dots, x_n]\}$$

$$m_{(a_1,\dots,a_n)} := \text{ the kernel of "evaluation at } (a_1, \dots, a_n)\text{"}$$

is a bijection (and note that $m_{(a_1,\dots,a_n)} = \langle x_1 - a_1, \dots, x_n - a_n \rangle$).

Corollary 1.2: If $I \subset \mathbb{C}[x_1, \dots, x_n]$ is an ideal, then the restriction:

$$m|_{V(I)} : V(I) \to \{\text{maximal ideals containing } I\} \text{ is a bijection}$$

Corollary 1.3: The natural "closure" on ideals in $\mathbb{C}[x_1, \dots, x_n]$:

$$I \mapsto \overline{I} := \bigcap_{x \in V(I)} \{\text{maximal ideals } I \subset m_x\}$$

coincides with the "radicalization:"

$$\overline{I} = \sqrt{I} := \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f^N \in I \text{ for some } N > 0\}$$

In particular, a prime ideal is equal to its closure (= radical).

Review the proofs of the Nullstellensatz and Corollary 1.3.

Definition: Let $S \subset \mathbb{C}^n$ be an arbitrary subset. Then:

$$I(S) := \bigcap_{x \in S} m_x$$

is the *ideal* in $\mathbb{C}[x_1, \ldots, x_n]$ of polynomials that vanish on S.

Definition: An algebraic subset $V(I) \subset \mathbb{C}^n$ is *irreducible* if:

$$V = V_1 \cup V_2 \Rightarrow V = V_1 \text{ or } V = V_2$$

whenever $V_1 = V(I_1)$ and $V_2 = V(I_2)$ are algebraic sets.

Exercise 1.1: If $V(I)$ is an irreducible algebraic set, then:

$$\sqrt{I} = I(V(I))$$

is a *prime* ideal, and vice versa, $V(\mathcal{P}) \subset \mathbb{C}^n$ is an irreducible algebraic set whenever $\mathcal{P} \subset \mathbb{C}[x_1, \ldots, x_n]$ is a prime ideal.

Complex affine varieties are the irreducible algebraic sets in $\mathbb{C}^n$ (for some n). It is crucial to have an "intrinsic" description of an affine variety (i.e. without reference to the ambient $\mathbb{C}^n$), as a topological space equipped with a *sheaf of* $\mathbb{C}$*-algebras*. To this end, suppose:

$$X = V(\mathcal{P}) \subset \mathbb{C}^n$$

is an irreducible algebraic set. Then:

- The *coordinate ring* of X is $\mathbb{C}[X] := \mathbb{C}[x_1, \ldots, x_n]/\mathcal{P}$.
  ($\mathbb{C}[X]$ is an integral domain that is finitely generated as a $\mathbb{C}$-algebra.)
- The *field of rational functions* on X is the fraction field:
  $$\mathbb{C}(X) := \left\{ \frac{f}{g} \mid f, g \in \mathbb{C}[X] \text{ and } g \neq 0 \right\} / \left(\frac{f}{g} \sim \frac{f'}{g'} \text{ if } fg' = f'g \right)$$
  ($\mathbb{C}(X)$ has finite transcendence degree as a field extension of $\mathbb{C}$.)
- The *points* of X are in bijection with the maximal ideals in $\mathbb{C}[X]$:
  $$\{x \in X\} \leftrightarrow \{\text{maximal ideals } m_x \subset \mathbb{C}[X]\}$$
- The *Zariski topology* on X is generated by "open" sets of the form:
  $$U_f := \{x \in X \mid f \notin m_x\} \subset X \text{ for each } f \in \mathbb{C}[X]$$
  (U_f is the complement (in X) of the hypersurface $V(f) \subset \mathbb{C}^n$)
- The *germ of rational functions* at each $x \in X$ is:
  $$\mathcal{O}_x := \left\{ \frac{f}{g} \mid g \notin m_x \right\} \subset \mathbb{C}(X)$$
  ($\mathcal{O}_x$ has unique maximal ideal $\widetilde{m}_x := \left\{ \frac{f}{g} \mid f \in m_x, g \notin m_x \right\} \subset \mathcal{O}_x$.)

Note: I stopped writing the equivalence relation. So sue me.

- The *regular functions* on each open set $U \subset X$ are:

$$\mathcal{O}_X(U) := \bigcap_{x \in U} \mathcal{O}_x \subset \mathbb{C}(X)$$

($\mathcal{O}_X(U)$ is an integral domain containing $\mathbb{C}[X]$.)

Exercise 1.2. (a) Prove that the sets:

$$U_I := \{x \in X \mid I \not\subset m_x\} \text{ determined by ideals } I \subset \mathbb{C}[X]$$

are always finite unions of the $U_f \subset X$, and that these sets are precisely the open sets of the Zariski topology on X.

(b) Prove that

$$\mathcal{O}_X(U_f) = \left\{ \frac{g}{f^n} \mid n \geq 0, g \in \mathbb{C}[X] \right\} \subset \mathbb{C}(X)$$

In particular, conclude that $\mathcal{O}_X(X) = \mathbb{C}[X]$.

(c) Prove that each $\mathcal{O}_X(U)$ is finitely generated as a $\mathbb{C}$-algebra.

(d) Show that the germ of rational functions $\mathcal{O}_x$ at x satisfies

$$\mathcal{O}_x = \bigcup_{x \in U} \mathcal{O}_X(U)$$

and is **not** usually finitely generated as a $\mathbb{C}$-algebra.

(e) Prove that

$$\mathbb{C}(X) = \cup_{x \in X} \mathcal{O}_x = \cup_{U \subset X} \mathcal{O}_X(U)$$

Note also that $U \subset V \Rightarrow \mathcal{O}_X(V) \subset \mathcal{O}_X(U)$.

Exercise/Example 1.3: Describe all the data above for $X = \mathbb{C}^1$ and $\mathbb{C}^2$. In particular, show that:

$$\mathcal{O}_{\mathbb{C}^2}(\mathbb{C}^2 - \{0\}) = \mathbb{C}[x_1, x_2] = \mathcal{O}_{\mathbb{C}^2}(\mathbb{C}^2)$$

showing that shrinking an open set does not always increase the ring of regular functions.

Next, prove that the Cartesian product of open sets in $\mathbb{C}^1$ is an open set in $\mathbb{C}^2$, but that, for example, the open subset:

$$U_{x_1 - x_2} \subset \mathbb{C}^2 \text{ (the complement of the diagonal)}$$

does **not** contain *any* (non-empty) product of open sets in $\mathbb{C}^1$. Thus, in particular, the Zariski topology on $\mathbb{C}^2$ is not the product topology of the two Zariski topologies on $\mathbb{C}^1$.