Complex Algebraic Geometry: Smooth Curves

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12. First Steps Towards Classifying Curves. The Riemann-Roch Theorem is a powerful tool for classifying smooth projective curves, i.e. giving us a start on the following questions:

"What are all the curves of a given genus (up to isomorphism)?" or

"When is there a smooth curve of genus g and degree d in $\mathbb{P}^r$?"

Genus Zero Curves: Abstractly, they are easy to describe:

Proposition 12.1. If $g(C) = 0$, then C is isomorphic to $\mathbb{P}^1$.

Proof: Consider $D = p$. Then by the Riemann-Roch *inequality*:

$$l(p) = \dim(L(p)) \geq 1 + 1 - g = 2$$

so there is a non-constant $\phi \in \mathbb{C}(C)$ with pole of order one at p, and no other poles, defining a regular map: $\Phi : C \to \mathbb{P}^1$ of degree one which is therefore an isomorphism.

The *rational normal curve* is the embedding:

$$\Phi : \mathbb{P}^1 \to \mathbb{P}^d; \quad (x : y) \mapsto (x^d : x^{d-1}y : x^{d-2}y^2 : \cdots : y^d)$$

and its image under arbitrary change of basis of $\mathbb{P}^d$.

Notice that every map of degree d from $\mathbb{P}^1$ to $\mathbb{P}^r$ whose image does not lie in any hyperplane is a projection of the rational normal curve. There is a sort of converse to this.

Definition. $C \subset \mathbb{P}^r$ *spans* $\mathbb{P}^r$ if it is not contained in a hyperplane.

Proposition 12.2: Every $C \subset \mathbb{P}^d$ of degree less than d **fails** to span. The only curve of degree d that spans $\mathbb{P}^d$ is the rational normal curve.

Proof: Since $l(D) \leq d + 1$ for all divisors on all curves, the first sentence is immediate. As for the second, suppose $l(D) = d+1$, choose $p \in C$ and note that $l(D - (d-1)p) = 2$. As in Proposition 12.1, this implies that $C = \mathbb{P}^1$, and then embedding is the rational normal curve because a projection would have larger degree.

The Riemann-Roch Theorem is very useful for finding embeddings of smooth curves of higher genus by means of:

Definition/Exercise 12.1: Let D be an effective divisor.
Then the linear series $|D|$ is:

(a) *Base point free* if

$$l(D - p) = l(D) - 1 \text{ for all } p \in C$$

in which case the regular map: $\Phi : C \to \mathbb{P}^r; x \mapsto (\phi_1(x) : \cdots : \phi_r(x) : 1)$ (for a choice of basis $\phi_1, \cdots \phi_r, 1 \in L(D)$) has degree d. In fact, the map can be defined without resorting to a choice of basis if $\mathbb{P}^r$ is replaced by the *dual* $|D|^\vee$ of the projective space $|D|$.

(b) *Very ample* if

$$l(D-p-q) = l(D) - 2 \text{ for all } p, q \text{ (including } p = q) \text{ in } C$$

in which case the map Φ from (a) is a *closed embedding*.

Genus One Curves: These curves are distinguished in that:

$$l(K_C) = \dim(\Omega[C]) = 1, \text{ and } \deg(K_C) = 0$$

so they admit differential forms ω with no zeroes or poles. This is also a consequence of the fact that genus one curves (with a choice of origin) are (Lie) groups. This was proved classically with the help of:

Proposition 12.3. All curves of genus one are smooth plane cubics.

Proof: Consider the linear series $L(np)$ for $p \in C$:

(a) $l(p) = 1$, so $L(p) = \mathbb{C}$, the constant functions.

(b) $l(2p) = 2$. Let $\phi \in L(2p) - L(p)$. This defines a two-to-one map:

$$\Phi : C \to \mathbb{P}^1;\ x \mapsto (\phi(x) : 1)$$

ramifying over ∞ and three other points (by Riemann-Hurwitz).

(c) $l(3p) = 3$. Let $\psi \in L(3p) - L(2p)$. Then:

$$\Phi : C \to \mathbb{P}^2; x \mapsto (\phi(x) : \psi(x) : 1)$$

is a closed embedding as a smooth plane curve.

(d) $l(4p) = 4$. This has basis $1, \phi, \psi, \phi^2$.

(e) $l(5p) = 5$. This has basis $1, \phi, \psi, \phi^2, \phi\psi$.

(f) $l(6p) = 6$. There is a linear dependence involving ψ^2 and ϕ^3:

$$\psi^2 - k\phi^3 = a\phi\psi + b\phi^2 + c\psi + d\phi + e$$

This is the equation defining the image of C in (c). Indeed, after completing the square, it has the form:

$$y^2 = (x-a)(x-b)(x-c)$$

where $a, b, c \in \mathbb{C}$ are the (distinct) points over which the map in (b) ramifies. Moreover, after composing with an automorphism of $\mathbb{P}^1$, we may assume that:

$$a = 0,\ b = 1,\ c = \lambda$$

for some $\lambda \neq 0, 1, \infty$.

Curves of Higher Genus. These curves break into two camps; the hyperelliptic curves and the *canonical* curves embedded in $\mathbb{P}^{g-1}$ by the linear series $|K_C|$. For the first few "higher" genera, the canonical curves are easy to describe. After that, things are more subtle.

Definition. A curve C of genus ≥ 2 is *hyperelliptic* if there is a map:

$$\Phi : C \to \mathbb{P}^1 \text{ of degree } 2$$

or, equivalently, if there exist $p, q \in C$ such that $l(p+q) = 2$.

Proposition 12.4. Every curve of genus 2 is canonically hyperelliptic.

Proof: The canonical divisor satisfiess:

$$\deg(K_C) = 2g - 2 = 2 \text{ and } l(K_C) = 2$$

Thus this hyperelliptic map is canonical in the sense that it is the map to $\mathbb{P}^1$ (or $|K_C|^\vee$) induced by the canonical linear series. It is also canonical in the sense that it is the unique degree two map to $\mathbb{P}^1$, for either of the following two reasons:

Proposition 12.5. For divisors of degree $2g-2$ on a smooth curve C of genus g, either $D = K_C$ or else $l(D) = g - 1$.

Proof: Suppose $\deg(D) = 2g - 2$. Then by Riemann-Roch:

$$l(D) - l(K_C - D) = g - 1$$

But $l(K_C - D) = 0$ unless $D = K_C$.

Proposition 12.6. There is *at most* one map of degree two from a curve C of genus ≥ 2 to $\mathbb{P}^1$ (modulo automorphisms of $\mathbb{P}^1$).

Proof: Suppose there were two such maps: Φ and $\Psi : C \to \mathbb{P}^1$. Choose a point $p \in C$, which, for convenience, is not a ramification point of either map. Let $p + q = \Phi^{-1}(\Phi(p))$ and $p + r = \Psi^{-1}(\Psi(p))$. Then we conclude that there are rational functions:

$$\phi \in L(p+q) - \mathbb{C} \text{ and } \psi \in L(p+r) - \mathbb{C}$$

and we can further conclude that $1, \phi, \psi \in L(p+q+r)$ are linearly independent. This would define a regular map $\Xi : C \to \mathbb{P}^2$ of degree 3, which either embeds C as a smooth plane cubic (in which case $g = 1$) or else maps C onto a nodal or cuspidal cubic curve (in which case $g = 0$).

Proposition 12.7. For divisors D of degree $d \geq 2g+1$ on C,

$$\Phi : C \to \mathbb{P}^{d-g} = |D|^\vee$$

is a closed embedding.

Proof: By Riemann-Roch, we have:

$l(D) = d - g + 1,\ l(D-p) = d - g$ and $l(D-p-q) = d - 1 - g$

since $l(K_C - D) = l(K_C - D + p) = l(K_C - D + p + q) = 0$.

Similarly,

Proposition 12.8. If $D = K_C$ and C is not hyperelliptic, then:

$$\Phi : C \to \mathbb{P}^{g-1} = |K_C|^\vee$$

is a closed embedding, and conversely, if $C \subset \mathbb{P}^{g-1}$ is a genus g curve of degree $2g-2$ that spans $\mathbb{P}^{g-1}$, then C is a canonically embedded (non-hyperelliptic) curve.

Proof: As in Proposition 12.7, the embedding follows from:

$$l(0) = l(p) = l(p+q) = 1$$

for non-hyperelliptic curves. From $l(p+q) = 2$ for selected $p, q \in C$ on a hyperelliptic curve, it follows that $|K_C|$ does not embed such curves (see the Proof of Proposition 12.9 below to see what does happen). From Proposition 12.5, we conclude that all spanning smooth curves of genus g and degree $2g-2$ are canonically embedded.

Proposition 12.9. There exist hyperelliptic curves of every genus.

Proof: The affine plane curve $C \subset \mathbb{C}^2$ defined by:

$$y^2 = (x - a_1) \cdots (x - a_{2g+1})$$

and mapping to $\mathbb{C}^1$ via projection on the x-axis should be completed, by adding one point at infinity, to a smooth projective curve of genus g. The closure in $\mathbb{P}^2$ won't serve the purpose, since it is singular for $g \geq 2$. Instead, let $p \in C$ be one of the ramification points, and suppose there were a smooth, projective curve $C \subset \overline{C}$ obtained by adding one point. Then:

$$\phi \in L(2p)$$

would be a rational function with pole of order two at p, and by the Riemann-Roch theorem, there would be a basis:

$$1, \phi, \phi^2, \cdots, \phi^{g-1} \in L((2g-2)p)$$

from which it follows (by Proposition 12.5) that $(2g-2)p = K_C(!)$ and the canonical map for a hyperelliptic curve is the 2:1 map to $\mathbb{P}^1$, followed by the embedding of $\mathbb{P}^1$ in $\mathbb{P}^{g-1}$ as a rational normal curve. We may further extend to a basis:

$$1, \phi, \phi^2, \cdots, \phi^g, \psi \in L((2g+1)p)$$

by introducing a new rational function $\psi \in L((2g+1)p) - L((2g)p)$. This would give an embedding (by Proposition 12.7):

$$\Phi : \overline{C} \hookrightarrow \mathbb{P}^{g+1}$$

After completing the square (as in Proposition 12.3), one would have $\overline{C} \subset \mathbb{P}^{g+1}$ (embedded by $x \mapsto (\phi(x) : \phi^2(x) : \cdots : \phi^g(x) : \psi(x) : 1)$) as the closure of

$$C = V(x_2 - x_1^2, x_3 - x_1^3, \cdots, x_g - x_1^g, x_{g+1}^2 - \prod(x_1 - a_i)) \subset \mathbb{C}^{g+1}$$

Exercise: Prove that the closure of C is smooth, by finding enough homogeneous polynomials in $I(\overline{C}) \subset \mathbb{C}[x_1, \ldots, x_{g+2}]$.

Proposition 12.10. The following projective curves are embedded by the canonical linear series, hence in particular are not hyperelliptic:

(a) (Genus 3) A smooth plane curve of degree four.

(b) (Genus 4) A smooth complete intersection $Q \cap S \subset \mathbb{P}^3$ of surfaces of degrees two and three.

(c) (Genus 5) A smooth complete intersection $Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^4$ of hypersurfaces of degree two.

Proof: Each is an embedded curve of genus g and degree $2g-2$, as can be checked from the Hilbert polynomial. Therefore, each is an embedded canonical curve

Exercise: Prove that every smooth plane curve of degree ≥ 4 is not hyperelliptic by finding an embedding of the curve (of genus $g = \binom{d-1}{2}$) in $\mathbb{P}^{g-1}$ of degree $2g-2$.

Theorem 12.11. "Most" curves of genus ≥ 3 are not hyperelliptic.

Sketch of a Proof: As shown in Propositions 12.6 and 12.9, each such curve has a unique degree two map to $\mathbb{P}^1$ (up to automorphisms) ramified at $2g+2$ distinct (unordered) points:

$$a_1, \cdots, a_{2g+2} \in \mathbb{P}^1$$

This means that we can think of the set of **all** hyperelliptic curves as a quotient of the parameter space:

$$\mathbb{P}^{2g+2} - \Delta$$

of unordered distinct $2g+2$-tuples of points on $\mathbb{P}^1$, by the relation:

$$(a_1 + \cdots + a_{2g+2}) \sim (a_1' + \cdots a_{2g+2}') \Leftrightarrow \alpha(a_i) = a_i'$$

for some $\alpha \in \mathrm{PGL}(2, \mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1)$. It follows from *invariant theory* that there is a **quasi-projective** variety $\mathcal{H}_g$ and a surjective map:

$$h_g : (\mathbb{P}^{2g+2} - \Delta) \to \mathcal{H}_g$$

such that the fibers of h_g are the equivalence classes of sets of points. In other words, $\mathcal{H}_g$ is a (coarse) *moduli space* for hyperelliptic curves of genus g, and the dimension of $\mathcal{H}_g$ is $2g+2-\dim(\mathrm{PGL}(2, \mathbb{C})) = 2g-1$.

On the other hand, recall from Proposition 12.7 that every smooth curve can be embedded in $\mathbb{P}^{g+1}$ by simply choosing a divisor of degree $2g+1$ on the curve, and from Proposition 12.9 that hyperelliptic curves can be explicitly so embedded. Suppose $C \subset \mathbb{P}^{g+1}$ is such an embedded curve, of degree $2g+1$, and consider a generic projection to $\mathbb{P}^1$:

(†) $\Phi : C \to \mathbb{P}^1$ of degree $2g+1$ ramifiying over $a_1, \ldots, a_{6g} \in \mathbb{P}^1$

(the number of ramification points is computed by Riemann-Hurwitz). One can associate a "monodromy representation" of the fundamental group of $\mathbb{P}^1 - \{a_1, \ldots, a_{6g}\}$ by choosing a base point $a_0 \in \mathbb{P}^1$, and following the $2g+1$ sheets of the cover over loops γ_i emanating from a_0 and looping once around a_i to get transpositions of the sheets and a representation into the symmetric group:

$$(*)\ \rho : \pi_1(\mathbb{P}^1 - \{a_1, \cdots, a_{6g}\}, a_0) \to \Sigma_{2g+1};\ \gamma_i \mapsto t_i, \prod_{i=1}^{6g} t_i = 1, \langle t_i \rangle = \Sigma_{g+1}$$

Four big theorems are needed:

Fundamental Theorem of Riemann Surfaces.
Each representation $(*)$ comes from a uniquely determined map (†).

Irreducibility Theorem. There is a *variety* Hu_g, the Hurwitz variety, parametrizing the covers $(*)$, together with a finite surjective map:

$$\pi_g : \mathrm{Hu}_g \to (\mathbb{P}^{6g} - \Delta)$$

of degree equal to the number of ways of writing:

$1 = t_1 \cdots t_{6g}$ as a product of transpositions that generate Σ_{2g+1}

Fundamental Theorem on Moduli of Curves: There is a quasi-projective variety $\mathcal{M}_g$ and a surjective regular map:

$$m_g : \mathrm{Hu}_g \to \mathcal{M}_g$$

whose fibers are the equivalence classes under the relation:

$$[C \to \mathbb{P}^1] \sim [C' \to \mathbb{P}^1] \Leftrightarrow C \cong C'$$

Deformation Theory: The Zariski tangent space to each fiber:

$$m_g^{-1}(C) = \{f : C \to \mathbb{P}^1 \mid f \text{ has simple ramification}\}$$

at $[f]$ is $L(D)$, where $D = f^{-1}(-K_{\mathbb{P}^1})$ (with multiplicities, if necessary) is a divisor of degree $2(2g+1)$, and so $l(D) = 3g+3$ by Riemann-Roch.

Thus,

$$\dim(\mathcal{M}_g) = \dim(\mathrm{Hu}_g) - \dim(m_g^{-1}(C)) = 6g - (3g+3) = 3g - 3$$

and so $\dim(\mathcal{M}_g) > \dim(\mathcal{H}_g)$ when $g \geq 3$, which gives meaning to the assertion that "most" curves are not hyperelliptic.

Example. The smooth plane curves $C \subset \mathbb{P}^2$ of degree 4 are parametrized by an open subset $U \subset \mathbb{P}^{14}$, since $\dim \mathbb{C}[x, y, z]_4 = 15$, and two such curves are isomorphic if and only if they are related by a change of basis of $\mathbb{P}^2$. In fact, there is an open $W \subset \mathcal{M}_3$ and a surjective map:

$$U \to W \subset \mathcal{M}_3 \text{ with fibers } \mathrm{PGL}(3, \mathbb{C})$$

and the dimensions work out: $14 - \dim(\mathrm{PGL}(3, \mathbb{C}) = 6 = 3(3) - 3$. Note that $\mathcal{M}_3 - W = \mathcal{H}_3$, which has dimension $2(3) - 1 = 5$.