Categories, Symmetry and Manifolds

Math 4800, Fall 2020

7. G-Linear Maps. We turn our attention now to morphisms of representations, i.e. to G-linear maps of vector spaces equipped with actions of the group G.

Let (V, ρ) and (W, τ) be representations of a group G, and let:

$$\hom_G(V, W) \subset \hom(V, W)$$

be the vector subspace of G-linear maps $f : V \to W$.

Proposition 7.1. The kernel, image and cokernel of a G-linear map $f \in \hom_G(V, W)$ are also representations of G.

Proof. The kernel $K = \ker(f)$ is an invariant subspace of V since if $f(v) = 0$, then $f(gv) = gf(v) = g0 = 0$. Similarly, the image $I = f(V)$ is an invariant subspace of W, since if $w = f(v)$ for some $v \in V$, then $gw = f(gv)$. Finally, the cokernel is:

$$W/I = \{w + f(V)\} \text{ and } g(w + f(V)) = gw + f(V)$$

is a well-defined action of G on the coset vector space W/I since I is invariant. $\square$

Example. (a) Let $V = U \oplus W$ be a direct sum of (complementary) G-representations, and consider the G-linear inclusion map $i : U \to V$. Then the map:

$$q : W \to V/U, \;\; q(w) = w + U$$

is an isomorphism and G-linear. We say that this quotient "lifts" to the sub-representation $W \subset V$ via the isomorphism $q^{-1} : V/U \to W$ followed by $W \subset V$.

(b) Consider the trivial sub-representation $\langle e_1 \rangle$ of the representation

$$\rho(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \text{ of } (\mathbb{C}, +, 0) \text{ on } \mathbb{C}^2$$

Then $\mathbb{C}^2/\langle e_1 \rangle$ is also trivial since $z \cdot (e_2 + \langle e_1 \rangle) = (ze_1 + e_2) + \langle e_1 \rangle = e_2 + \langle e_1 \rangle$. Thus in this case both the invariant subspace $\langle e_1 \rangle$ and the quotient space are trivial representations, even though the overall representation is not trivial. In particular, this quotient representation does not lift to a sub-representation of $\mathbb{C}^2$ since if it did, the overall representation would be trivial.

Schur's Lemma 7.2. If V and W are irreducible complex representations of G, then either $\hom_G(V, W) = 0$ or else $\hom_G(V, W)$ is a one-dimensional vector space of isomorphisms (with the exception of 0, which is evidently not an isomorphism).

Proof. Let $f : V \to W$ be G-linear. Then the kernel of f is G-invariant, so $\ker(f) = 0$ or $\ker(f) = V$, since V is an irreducible representation. Similarly, the image of f is G-invariant, so $\mathrm{im}(f) = 0$ or $\mathrm{im}(f) = W$. Putting these together, we see that either $f = 0$ or else f is both injective and surjective, i.e. an isomorphism. Now suppose $h : V \to W$ is another G-linear isomorphism. Then consider the symmetry $\sigma = f^{-1} \circ h : V \to V$ and the G-linear maps:

$$\sigma_\lambda = \sigma - \lambda \cdot \mathrm{id}_V : V \to V \text{ for } \lambda \in \mathbb{C}$$

Each of these is either 0 or an isomorphism because it is a G-linear map of irreducible representations, but if λ is an **eigenvalue** of σ, then σ_λ has a non-trivial kernel, so it cannot be an isomorphism, and thus $\sigma_\lambda = 0$, which gives $f^{-1} \circ h = \lambda \cdot \mathrm{id}_V$ and $\lambda \cdot f = h$. Thus every G-linear map is a multiple of f, i.e. $\dim(\hom_G(V, W)) = 1$. $\square$

Corollary 7.3. Suppose V is a completely reducible complex representation of G. That is, suppose:

$$V = U_1 \oplus \cdots \oplus U_m$$

is a direct sum of irreducible representations $U_1, ..., U_m$ (possibly with repetitions). Then $\hom_G(V, U) \neq 0$ if and only if $U \cong U_i$ for some i

Proof. If U is not isomorphic to any of the U_i and $h \in \hom_G(V, U)$, then each of the restricted maps: $h|_{U_i} : U_i \to U$ is the zero map by Schur's lemma, and so:

$$h(v) = h(u_1 + \cdots + u_m) = h(u_1) + \cdots + h(u_m) = 0$$

for all $v = u_1 + \cdots + u_m \in V$. In other words, h itself is the zero map. □

Given a G-representation U, let

$$nU = U \oplus U \oplus \cdots \oplus U \ (n \text{ times})$$

Corollary 7.4. Let V as in Corollary 7.3 be written as:

$$V = n_1 U_1 \oplus \cdots \oplus n_l U_l$$

collecting the isomorphic representations. If $U \cong U_i$, then $\dim(\hom_G(V, U_i)) = n_i$.

Proof. This uses the second part of Schur's lemma. If $U \cong U_i$, let $f : U_i \to U$ be an isomorphism (uniquely determined up to the choice of scalar by Schur's Lemma). Then any G-linear map $h : V \to U$ restricts to:

$$h|_{n_i U_i} : U_i \oplus \cdots \oplus U_i \to U$$

and to a multiple of f when further restricted to each of the n_i copies of U_i. Thus:

$$h|_{n_i U_i}(u_1 + \cdots + u_{n_i}) = \lambda_1 f(u_1) + \cdots + \lambda_{n_i} f(u_{n_i})$$

and conversely, each choice of vector $(\lambda_1, ..., \lambda_{n_i}) \in \mathbb{C}^{n_i}$ determines a unique G-linear map $h : V \to U$ that is $\lambda_1 f + \cdots + \lambda_{n_1} f$ when restricted to $n_i U_i$. □

The Regular Representation. Let G be a finite group and define:

$$\mathbb{C}[G] = \langle e_g \mid g \in G \rangle$$

to be the complex vector space $\mathbb{C}^{|G|}$ with one basis vector for each element $g \in G$. This vector spaces comes equipped with the "regular" representation of G given by:

$$\rho_{reg}(h)(e_g) = e_{hg}$$

i.e. $\rho_{reg}(h)$ permutes the basis vectors by left translation by the element $h \in G$.

Example. If $G = C_n$ is the cyclic group, then $\mathbb{C}[C_n] = \langle e_x, e_{x^2}, \ldots, e_{x^n} = e_1 \rangle$ and under the regular representation,

$$\rho_{reg}(x)(e_{x^i}) = e_{x^{i+1}} \text{ for } i < n \text{ and } \rho_{reg}(x)(e_{x^n}) = e_x$$

In other words, this is the cyclic representation of C_n from §6, which we decomposed:

$$\mathbb{C}^n = \oplus_{m=0}^{n-1} \chi_m$$

into a direct sum of n characters (one-dimensional representations).

We can think of the regular representation as the vector space of **functions**:

$$f : G \to \mathbb{C}$$

from G to the complex numbers, in which e_g are the "delta functions"

$$e_g(h) = 0 \text{ if } h \neq g \text{ and } e_g(g) = 1$$

An arbitrary function $f : G \to \mathbb{C}$ is then a sum:

$$f = \sum_{g \in G} f(g) e_g$$

of f in terms of delta functions. From this point of view, the regular representation is the action of the group G on the vector space of functions taking:

$$h \cdot f = f \circ \text{left translation of } G \text{ by } h^{-1}$$

which is exactly the content of: $h \cdot e_g = e_{hg}$. (Note the inverse!)

Thinking of elements of $\mathbb{C}[G]$ as functions may or may not help when thinking about the regular representation of finite groups, but it gives us a road map for what to do with some groups (e.g. orthogonal or unitary groups) that are not finite. The idea is to limit the space of all functions $f : G \to \mathbb{C}$ (which is too big) to more manageable G-invariant subspaces. These will not be finite-dimensional, since in fact orthogonal and unitary groups have infinitely many irreducible complex representations, but when we do this right, it will help us to find the countably many complex representations.

Theorem 7.5. The regular representation $\mathbb{C}[G]$ of G satisfies:

$$\hom_G(\mathbb{C}[G], U) = U \text{ via the map } h \mapsto h(e_{id})$$

for **all** irreducible complex representations U of the group G. Thus in particular,

$$\dim(\hom_G(\mathbb{C}[G], U)) = \dim(U)$$

Proof. Let $u \in U$, and define a linear map $h_u : \mathbb{C}[G] \to U$ by setting:

$$h_u(e_g) = g \cdot u \text{ for each basis vector } e_g$$

where $g \cdot u \in U$ is the result of acting u by the group element g. In other words:

$$h_u(\sum_{g \in G} \lambda_g e_g) = \sum_{g \in G} \lambda_g g \cdot u \text{ for each vector } v = \sum_{g \in G} \lambda_g e_g \in \mathbb{C}[G]$$

Then for all $g' \in G$ and all basis vectors e_g of $\mathbb{C}[G]$, we have:

$$h_u(g' \cdot e_g) = h_u(e_{g'g}) = (g'g)u = g'(gu) = g' h_u(g)$$

so h_u is a G-linear map! Moreover, we see that: $h_u(e_{id}) = id \cdot u = u$ so h_u maps to u in the statement of the Theorem. If $h : \mathbb{C}[G] \to U$ is **any** G-linear map with $h(e_{id}) = u$, then $h(e_g) = h(g \cdot e_{id}) = g \cdot h(e_{id}) = g \cdot u$, so $h = h_u$. This completes the proof of the Theorem! $\square$

Putting Corollary 7.4 together with Theorem 7.5, we get:

Corollary 7.6. The regular representation of a finite group G satisfies:

$$\mathbb{C}[G] = \bigoplus_{\text{irreducible } U} \dim(U) U$$

This gives us two fairly surprising results.

(1) Every irreducible representation of G is a summand of $\mathbb{C}[G]$. Even more:

(2) The dimensions of the irreducible representations U of G satisfy:

$$\dim(\mathbb{C}[G]) = |G| = \sum_U \dim(U)^2$$

Examples. (a) The only irreducible representations of an abelian group G are (one-dimensional) characters, so: $|G| = 1^2 + 1^2 + \cdots + 1^2$ and there must be $|G|$ characters. We've seen this already for the cyclic group C_n, where we found the n characters $\chi_0, ..., \chi_{n-1}$. Now we know those are the only ones.

(b) We have found three irreducible representations of $S_3 = D_6$:

- The trivial representation (dimension one)
- The sign character (also dimension one)
- The symmetries of the equilateral triangle (dimension two)

Since $6 = 1^2 + 1^2 + 2^2$, there are no others irreducible representations!

(c) We have found two irreducible representations of D_8:

- The trivial representation (dimension one)
- The symmetries of the square (dimension two)

Since $8 = 1^2 + 2^2 + 3$ and 3 is only a sum of squares in one way ($3 = 1^2 + 1^2 + 1^2$), there are three other characters. Let's find them. The subgroup

$$H = \{1, x^2\} \subset D_8 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$$

is **normal** since $x(x^2)x^{-1} = x^2$ and $y(x^2)y^{-1} = x^{-2}yy^{-1} = x^2$, and the quotient is the group of left cosets:

$$D_8/H = \{H, xH.yH, xyH\}$$

Since $x^2 \in H$, $y^2 = 1$ and $(xy)^2 = 1$, each of the cosets $\{xH, yH, xyH\}$ has order two in D_8/H, and so D_8/H is the Klein group $C_2 \times C_2$. This is an abelian group, so it has four characters (see (a)). Each of them determines a character of D_8 via:

$$D_8 \to D_8/H \to \mathbb{C}^*$$

These are the four characters (including the trivial character) of D_8.

Remark. If $H \lhd G$ then each irreducible representation of G/H is an irreducible representation of G via $G \to G/H \to \mathrm{Aut}(\mathbb{C}^n)$.

(d) In addition to the irreducible three-dimensional representation of A_4 as the symmetries of the tetrahedron, we can use the normal Klein subgroup:

$$K_4 = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \lhd A_4$$

to find three characters of A_4 via the map $A_3 \to A_4/K_4 = C_3$, and $12 = 9+1+1+1$, so we know these are all of them. The cosets making up A_4/K_4 are:

$$\{K_4,\ (1\ 2\ 3)K_4,\ (1\ 3\ 2)K_4\}$$

which quite clearly shows that this is the cyclic group C_3.

The notion of a *generalized* character of a representation will help us to be more systematic about finding the irreducible representations of a finite group. Suppose $\rho : G \to \mathrm{Aut}(V)$ is a G-representation, and consider the map:

$$\chi_\rho : G \to \mathbb{C} \text{ given by } \chi_\rho(g) = \mathrm{tr}(\rho(g))$$

i.e. $\chi_\rho(g)$ is the trace of the matrix associated to $\rho(g)$. If $\chi : G \to \mathbb{C}^*$ is itself a (one-dimensional) character, then $\chi_\chi = \chi$, since in this case the trace is just the complex number itself! For G-representations of dimension two or more, the character retains a lot of important information about the representaton.

One of the values of a generalized character is easy to compute:

Proposition 7.7. If (V, ρ) is a G-representation, then:

$$\chi_\rho(\mathrm{id}) = \dim(V)$$

Proof. The trace of the identity map is the dimension of the vector space! $\square$

Example. The character of the two-dimesional irreducible representation of S_3 described in Example 2 of §6 consists of six numbers:

$$\begin{array}{lclcl}
\chi_\rho(\mathrm{id}) & = & \mathrm{tr}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & = & 2 \\
\chi_\rho(1\ 2) & = & \mathrm{tr}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} & = & 0 \\
\chi_\rho(2\ 3) & = & \mathrm{tr}\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} & = & 0 \\
\chi_\rho(1\ 3) & = & \mathrm{tr}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} & = & 0 \\
\chi_\rho(1\ 2\ 3) & = & \mathrm{tr}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} & = & -1 \\
\chi_\rho(1\ 3\ 2) & = & \mathrm{tr}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} & = & -1
\end{array}$$

We see in particular that a generalized character can take the value 0. In fact:

(b) The character of the regular representation $\mathbb{C}[G]$ is:

$$\chi_\rho(\mathrm{id}) = |G| \text{ and } \chi_\rho(g) = 0 \text{ for } \textbf{all } g \neq \mathrm{id}$$

The former is Proposition 7.7. To see the latter, note that:

$$\rho(g)(e_h) = e_{gh} \neq e_h \text{ if } g \neq \mathrm{id}$$

and so the matrix for the action of g is a permutation matrix with no fixed basis vectors, i.e. there are only zeroes on the diagonal, so the trace is zero.

Definition 7.8. A function $\alpha : G \to \mathbb{C}$ is a **class function** if $\alpha(h) = \alpha(ghg^{-1})$ for all $g, h \in G$, i.e. if α is constant on the conjugacy classes of G.

Examples. (a) Let $C_1, ..., C_r \subset G$ be the conjugacy classes of G. The functions:

$$\delta_i(g) = \begin{cases} 1 \text{ if } g \in C_i \\ 0 \text{ otherwise} \end{cases}$$

are step functions that form a basis for the vector space of class functions.

(b) Each the character $\chi_\rho : G \to \mathbb{C}$ of a representation ρ is a class function since

$$tr(\rho(g)\rho(h)\rho(g)^{-1}) = tr(\rho(h)) \text{ for all } g \text{ and } h$$

Theorem 7.8. Define an inner product on the space $Z[G]$ of class functions by:

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g)$$

Then the following *orthogonality relations hold* among the characters:

(i) The characters of irreducible representations of G are orthonormal and

(ii) The characters of irreps are an orthonormal **basis** of $Z[G]$.

Example. Before we prove this, let's see the Theorem in practice for $G = S_3$.

- The conjugacy classes of S_3 are:

$$C_1 = \{\text{id}\}, \text{C}_2 = \{(1\ 2), (2\ 3), (1\ 3)\}, \text{C}_3 = \{(1\ 2\ 3), (1\ 3\ 2)\}$$

- The irreducible representations of S_3 are:

  χ_{tr} (trivial), χ_{sgn}(sign), and the two-dimensional representation ρ above

We arrange the characters of these representations in a **character table**.

	id	(1 2)	(1 2 3)
	1	3	2
χ_{tr}	1	1	1
χ_{sgn}	1	-1	1
ρ	2	0	-1

- The first row is a list of representatives from each conjugacy class C_i
- The second row is a list of the sizes $|C_i|$ of each conjugacy class.
- The first column is a list of the irreducible representations
- The rest of the table computes the characters of the representations.

You may now check the orthogonality relations. For example:

$$(\chi_\rho, \chi_\rho) = \frac{1}{6}\left(2^2 + 2(-1)^2\right) = 1 \text{ and } (\chi_{tr}, \chi_{sgn}) = \frac{1}{6}\left(1 + 3 \cdot (-1) + 2 \cdot 1\right) = 0$$

Preliminary. The key idea in the proof is to notice that $\hom(V, W)$ is itself a G-representation whenever V and W are G-representations. The action of G is:

$$(g \cdot f)(v) = g \cdot f(g^{-1} \cdot v)$$

so that $(g \cdot f) : V \to W$ is the function that maps $(g \cdot f)(gv) = g \cdot f(v)$ It is a representation of G since $((hg) \cdot f)(hgv) = (hg) \cdot f(hgv) = h \cdot (g \cdot f)(gv) = (hg) \cdot f(v)$ and in particular $f \in \hom_G(V, W)$ if and only if $(g \cdot f) = f$ for **all** $g \in G$, i.e.

- $\hom_G(V, W)$ is the largest subspace of $\hom(V, W)$ on which G acts trivially

The character of $\hom(V, W)$ is the product:

$$\chi_{\hom(\text{V,W})}(g) = \chi_V(g^{-1}) \cdot \chi_W(g)$$

as one can check choosing bases of eigenvalues for $g^{-1} : V \to V$ and $g : W \to W$.

In addition, notice that the character of the representation $V \oplus W$ satisfies:

$$\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$$

so that, for example, $\chi_{\mathbb{C}[G]}(g) = \sum_U \dim(U) \cdot \chi_U(g)$.

Proof. This relies on a pair of averaging maps. Given a representation V, define:

$$p : V \to V \text{ by setting } p(v) = \frac{1}{|G|}\sum_{g\in G} g\cdot v. \text{ Then}$$

(a) $h \cdot p(v) = p(v)$, so the image of p consists of G-invariant vectors of V

(b) $p(v) = v$ for all G-invariant vectors in V.

Thus $p : V \to V_G \subset V$ is the *projection* onto the subspace of G-invariant vectors, and it follows that $tr(p) = \dim(V_G)$. On the other hand,

$$tr(p) = \frac{1}{|G|}\sum_{g\in G} tr(g\cdot) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)$$

is the average of the values of the character of V. When we apply this to the space $\hom(V, W)$ of maps between representations,

$$\dim(\hom_G(V, W)) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g^{-1})\chi_W(g)$$

If V and W are **irreducible** representations, then from Schur's Lemma we get:

$$\frac{1}{|G|}\sum_{g\in G} \chi_V(g^{-1})\chi_W(g) = \left\{ \begin{array}{l} 1 \text{ if } V = W \\ 0 \text{ if } V \not\cong W \end{array} \right.$$

which is very nearly (i) of the Theorem. In fact, it is (i) since the characters of representations of a finite group satisfy $\chi_V(g^{-1}) = \overline{\chi_V(g)}$. To see this, recall that each matrix $A = \rho(g)$ has finite order, hence all its eigenvalues are roots of unity, and roots of unity satisfy $\zeta^{-1} = \overline{\zeta}$. But then the trace of A is a sum $\sum \zeta_i$ of roots of unity, and the trace of A^{-1} is $\sum \zeta_i^{-1} = \sum \overline{\zeta}_i = \overline{tr(A)}$.

We prove (ii) with a weighted average function. For a class function $\alpha \in Z[G]$,

$$p_\alpha : V \to V \text{ is defined by } p_\alpha(v) = \frac{1}{|G|}\sum_{g\in G} \alpha(g)g\cdot v.$$

Then we claim that $p_\alpha \in \hom_G(V, V)$. Indeed,

$$p_\alpha(h\cdot v) = \frac{1}{|G|}\sum_{g\in G}\alpha(g)(gh)\cdot v = \frac{h}{|G|}\sum_{g\in G}\alpha(g)(h^{-1}gh)\cdot v = \frac{h}{|G|}\sum_{g\in G}\alpha(h^{-1}gh)(h^{-1}gh)\cdot v$$

since α is a class function, and then this is just a reordering of the sum, so:

$$\frac{h}{|G|}\sum_{g\in G}\alpha(h^{-1}gh)(h^{-1}gh)\cdot v = h\cdot p_\alpha(v)$$

If V is an irreducible representation, then $p_\alpha = \lambda \cdot \mathrm{id}_V$ by Schur's Lemma, and:

$$\lambda\cdot\dim(V) = tr(p_\alpha) = \frac{1}{|G|}\sum_g \alpha(g)\chi_V(g) = (\overline{\alpha}, \chi_V)$$

so that in particular, if $(\overline{\alpha}, \chi_V) = 0$, then $\lambda = 0$ and $p_\alpha = 0$.

If the characters of irreducible representations **fail** to be a basis for $Z[G]$, then there is a nonzero class function $\alpha \in Z[G]$ such that $\overline{\alpha}$ is *orthogonal* to all characters. That is $(\overline{\alpha}, \chi_V) = 0$ for all irreducible representations. But then by linearity,

$(\overline{\alpha}, \chi_V) = 0$ for **all** characters, and $p_\alpha = 0$ for all representations of G. If we apply this to the regular representation $\mathbb{C}[G]$, we get:

$$p_\alpha(e_{\text{id}}) = \frac{1}{|G|}\sum_g \alpha(g)e_g = 0$$

from which we conclude that $\alpha(g) = 0$ for all values of g. This is a contradiction. □

After doing all this work, let's have some fun.

Character table for A_4. The conjugacy classes of A_4 are:

$$C_1 = \{\text{id}\}, C_2 = \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

$$C_3 = \{(1\ 2\ 3), (1\ 3\ 4), (1\ 4\ 2), (2\ 4\ 3)\}, C_4 = \{(1\ 3\ 2), (1\ 4\ 3), (1\ 2\ 4), (2\ 4\ 3)\}$$

and we've seen that the irreducible representations of A_4 are $\chi_{tr}, \chi_\omega, \chi_{\omega^2}, \rho$ where $\omega = e^{2\pi i/3}$ and $\chi_\omega(1\ 2\ 3) = \omega$ and $\chi_{\omega^2}(1\ 2\ 3) = \omega^2$ and $\rho(g)$ is a symmetry of the tetrahedron in three-space. Then the character table of A_4 is:

	id	(1 2)(3 4)	(1 2 3)	(1 3 2)
	1	3	4	4
χ_{tr}	1	1	1	1
χ_ω	1	1	ω	ω^2
χ_{ω^2}	1	1	ω^2	ω
ρ	3	-1	0	0

where the traces of $\rho(g)$ are computed as follows:

(i) $\rho((1\ 2)(3\ 4))$ is a 180 degree rotation about an axis, with eigenvalues $1, -1, -1$.

(ii) $\rho(1\ 2\ 3)$ is a 120 degree rotation about an axis, with eigenvalues $1, \omega, \omega^2$.

and similarly for $\rho(1\ 3\ 2)$.

When we have an incomplete list of irreducible representations, there are various methods for filling in the table. If we are missing one representation, then its character (if not the representation itself) can be deduced from the orthogonality relations. Other methods for finding new representations include:

Multiplying by a One-dimesional Character. If (V, ρ) is a representation of G and χ is a one-dimensional character of G, then:

$$\rho'(g) = \chi(g) \cdot \rho(g) : V \to V$$

is another representation. If V is irreducible, then $\chi \cdot \rho$ is irreducible, and:

$$\chi'_\rho = \chi \cdot \chi_\rho$$

because multiplying a matrix by a scalar multiplies the trace by the same scalar. Then by the orthogonality relations, we know that:

$$(V, \rho') \cong (V, \rho) \text{ if and only if } \chi_{\rho'} = \chi_\rho$$

Thus, for example, the square of the sign character is always the trivial character, and if $\rho : S_3 \to \text{Aut}(\mathbb{C}^2)$ is the irreducible two-dimensional representation, then (reading from the character table)

$$\chi(\rho) = (2, 0, -1) \text{ and } \chi_{sgn} = (1, -1, 1), \text{ so } \chi_{\rho'} = (2, 0, -1)$$

and multiplying by the sign does not produce a new representation. However, in the following example, tensoring by the sign

Character Table for S_4. The five conjugacy classes of S_4 have representatives:

$$\text{id}, (1\ 2), (1\ 2\ 3), (1\ 2\ 3\ 4), (1\ 2)(3\ 4)$$

and via the quotient group $S_4/K_4 = S_3$ and the symmetries of the cube, we count the three irreducible representations of S_3 plus one among the irreducible representations of S_4. This gives the following partial character table:

	id	(1 2)	(1 2 3)	(1 2 3 4)	(1 2)(3 4)
	1	6	8	6	3
χ_{tr}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
ρ_{tri}	2	0	-1	0	2
ρ_{cub}	3	-1	0	1	-1
ρ	3				

The fact that the missing character ρ is three-dimensional follows from:

$$24 = 1^2 + 1^2 + 2^2 + 3^2 + \dim(\rho)^2$$

and from the orthogonality relations, we obtain the full last line of the table:

$$(3, 1, 0, -1, -1)$$

and this is indeed the character of the representation $\chi_{sgn} \cdot \rho_{cub}$.

Another interesting way of obtaining new representations is by:

Automorphisms of G. If $\sigma : G \to G$ is a symmetry (in the category of groups), and $\rho : G \to \text{Aut}(V)$ is a representation, then the **composition**:

$$\rho \circ \sigma : G \to G \to \text{Aut}(V)$$

is a representation. We've seen in §5 that **conjugating** by $g \in G$ is a symmetry:

$$\sigma_g(h) = ghg^{-1}$$

but these "inner" automorphisms of G do not change conjugacy classes, and thus do not change characters of representations. However, groups do on occasion have "outer" automorphisms that do change characters of representations.

Character Table for A_5. The five conjugacy classes of A_5 have representatives:

$$\text{id},\ (1\ 2)(3\ 4),\ (1\ 2\ 3), (1\ 2\ 3\ 4\ 5), (1\ 3\ 5\ 2\ 4)$$

and the outer automorphism σ obtained by conjugating A_5 by the odd permutation (2 3 5 4) exchanges the last two conjugacy classes (while fixing the others). Consider the representation ρ_{dod} of A_5 given by the action of A_5 on the dodecahedron. Then:

$\rho_{dod}((1\ 2)(3\ 4))$ is a rotation by π, with trace $1 + (-1) + (-1) = -1$.

$\rho_{dod}(1\ 2\ 3)$ is a rotation by $2\pi/3$ or $4\pi/3$, with trace $1 + \omega + \omega^2 = 0$.

$\rho_{dod}(1\ 2\ 3\ 4\ 5)$ is a rotation by $2m\pi/5$, since it is an element of order 5. If it is by $2\pi/5$ or $8\pi/5$, then the trace is $1+\tau+\tau^4 = \phi = (1+\sqrt{5})/2$, the golden mean, where $\tau = e^{2\pi i/5}$. If it is by $4\pi/5$ or $6\pi/5$, then the trace is $1+\tau^2+\tau^3 = (1-\sqrt{5})/2 = 1-\phi$. Whichever rotation is taken by $\rho_{dod}(1\ 2\ 3\ 4\ 5)$, the square $(1\ 3\ 5\ 2\ 4)$ is taken to a rotation with the opposite trace. So the character of ρ_{dod} is either:

$$(3, -1, 0, \phi, 1 - \phi) \text{ or } (3, -1, 0, 1 - \phi, \phi)$$

and indeed, composing with the symmetry σ takes one to the other.

Thus, we have three out of five rows of the character table for A_5:

	id	(1 2)(3 4)	(1 2 3)	(1 2 3 4 5)	(1 3 5 2 4)
	1	15	20	12	12
χ_{tr}	1	1	1	1	1
ρ	3	-1	0	ϕ	$1-\phi$
$\rho \circ \sigma$	3	-1	0	$1-\phi$	ϕ
ρ_1					
ρ_2					

and once again, the dimension count will tell us the other two dimensions:

$$60 = 1^2 + 3^2 + 3^2 + \dim(\rho_1)^2 + \dim(\rho_2)^2$$

from which it follows that $\dim(\rho_1) = 4$ and $\dim(\rho_2) = 5$ since:

$$41 = 16 + 25 \text{ is the unique way to express 41 as a sum of squares}$$

So we seek four and five dimensional irreducible representations of the group A_5. For the first, consider the permutation representation:

$$\rho_{perm}(\sigma)(e_i) = e_{\sigma(i)} \text{ for } V = \langle e_1, e_2, ..., e_5 \rangle$$

The trace of $\rho_{perm}(\sigma)$ is the number of elements fixed by σ, so:

$$\chi_{\rho_{perm}} = (5, 1, 2, 0, 0)$$

Is this the missing five-dimensional irreducible representation? No!

$$(\chi_{\rho_{perm}}, \chi_{\rho_{perm}}) = \frac{1}{60}\left(5^2 + 15(1^2) + 20(2^2)\right) = 2$$

But we knew is wasn't irreducible anyway, since:

$$e_1 + e_2 + e_3 + e_4 \text{ is an invariant vector}$$

from which it follows that the representation (V, ρ_{perm}) satisfies

$$V = U \oplus W \text{ where } U \text{ is the one-dimensional trivial representation}$$

This gives us W with $\chi_W = \chi_V - \chi_U = (4, 0, 1, -1, -1)$ and **this** is irreducible. This and orthogonality allows us to finish off the table:

	id	(1 2)(3 4)	(1 2 3)	(1 2 3 4 5)	(1 3 5 2 4)
	1	15	20	12	12
χ_{tr}	1	1	1	1	1
ρ	3	-1	0	ϕ	$1-\phi$
$\rho \circ \sigma$	3	-1	0	$1-\phi$	ϕ
W	4	0	1	-1	-1
ρ	5	1	-1	0	0

with the character of the missing representation. To find the representation itself, we will need to discuss tensor products of representations.