4800 - 20

Character Tables

Last time: G finite gp.

$$\mathbb{C}[G] = \langle e_g \mid g \in G \rangle = \{\textstyle\sum_{\mathbb{C}} z_g e_g\}$$

(basis vectors)

$$h \cdot e_g = e_{hg} \quad \text{defines the action}$$

$$\mathbb{C}[G] = \{\text{functions } f : G \to \mathbb{C}\}$$

$e_g \leftrightarrow$ delta functions $\delta_g(h) = \begin{cases} 1 & h = g \\ 0 & h \neq g \end{cases}$

$$f = \sum_{g \in G} \overset{\lambda_g}{\overset{\shortparallel}{f(g)}}\, \delta_g$$

$$h \cdot f = f \circ (\text{left mult. by } h^{-1})$$

$$\underline{h \cdot \delta_g} = \underline{\delta_{hg}}$$

$$\boxed{\mathbb{C}[G] = \bigoplus_{U \text{ irred.}} (\dim U) \cdot U}$$

<u>Problem</u>: <u>Find all the U's</u>

$$|G| = \sum_{U} (\dim U)^2$$

<u>Def</u>: The character χ_ρ of a representation (V, ρ) is:

$\chi_\rho : G \longrightarrow \mathbb{C}$ defined by

$$\chi_\rho(g) = \mathrm{tr}(\underline{\underline{\rho(g)}}).$$

Rmk: If $\rho = \chi : G \to \mathbb{C}^*$

is a one-dim'l repn, then

$$\chi_\chi(g) = tr(\chi(g)) = \chi(g).$$

i.e. $\chi_\chi = \chi$.

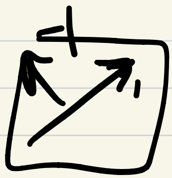
Example: Two-dim'l repn. of S_3

$$\rho : S_3 \longrightarrow \mathrm{Aut}(\mathbb{C}^2)$$

$$\rho(1\,2) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \rho(2\,3) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\chi_\rho(\mathrm{id}) = \mathrm{tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \quad (= \dim V)$$

$$\chi_\rho(\underline{1\,2}) = \mathrm{tr}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = 0$$



$$\chi_\rho(\underline{2\,3}) = 0, \quad \chi_\rho(\underline{1\,3}) = 0$$

$$\chi_\rho(1\,2\,3) = \mathrm{tr}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = -1$$

$$\chi_\rho(1\,3\,2) = \mathrm{tr}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \mathrm{tr}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= -1$$

Rmk: $\chi_\rho(h) = \chi_\rho(ghg^{-1})$

for all $g \in G$.

$\chi_\rho(h) = tr(\rho(h))$ $\quad B \cdot A \cdot B^{-1}$

$$\chi_\rho(ghg^{-1}) = tr(\rho(g)\cdot\rho(h)\cdot\rho(g)^{-1})$$

$$= tr(\rho(h)) = \chi_\rho(h).$$

Def: A function $f: G \to \mathbb{C}$ is a class function if $f(h) = f(ghg^{-1})$ for all $g, h \in G$.

Note: $\dim \{\text{class functions } f: G \to \mathbb{C}\}$

$= \#$ of conjugacy classes

E.g. χ_ρ $\qquad [h] = \{ghg^{-1}\}$
$= GhG^{-1}$

$$\chi_\rho(h) = \begin{cases} 2 & \text{if } [h] = [id] \\ 0 & \text{if } [h] = [(1\ 2)] \\ -1 & \text{if } [h] = [(1\ 2\ 3)] \end{cases}$$

__Thm:__ Define a Hermitian inner product of the space of class functions: $\alpha, \beta : G \to \mathbb{C}$ (constant on conj. classes)

$$\boxed{(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \cdot \overline{\beta(g)}}$$

Then the __characters__ of __irreps__ of G form an orthonormal basis for the class functions!!

S_3

	$[id]$	$[(12)]$	$[(123)]$	Conj. classes
	id	(12)	(123)	represent.
	1	3	2	# of els.
χ_{tr}	1	1	1	$\leftarrow f_1$
χ_{sgn}	1	-1	1	$\leftarrow f_2$
ρ	2	0	-1	$\leftarrow f_3$

irreps

$$\chi_\rho(g) = \operatorname{tr}(\rho(g))$$

$$(f_1, f_1) = \frac{1}{6} \sum_{g \in G} f_1^2(g) = \frac{1}{6}\left(1^2 + 3 \cdot 1^2 + 2 \cdot 1^2\right)$$
$$= 1.$$

$$(f_2, f_2) = \frac{1}{6}\left(1^2 + 3(-1)^2 + 2 \cdot 1^2\right) = 1$$

$$(f_3, f_3) = \frac{1}{6}\left(2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2\right) = 1$$

$\underline{C_3}$

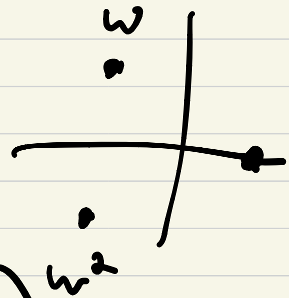
$[\omega^3 = 1]$

	id	x	x^2	
	1	1	1	
χ_0	1	1	1	$\leftarrow f_1$
χ_1	1	ω	ω^2	$\leftarrow f_2$
χ_2	1	ω^2	$\omega^4 = \omega$	$\leftarrow f_3$

$$(f_1, f_1) = \frac{1}{3}(1^2 + 1^2 + 1^2) = 1$$

$$(f_2, f_2) = \frac{1}{3}\left(1^2 + \omega \cdot \underbrace{\overline{\omega}}_{=\omega^2} + \omega^2 \cdot \underbrace{\overline{\omega^2}}_{=\omega}\right) = 1$$

$$(f_1, f_2) = \frac{1}{3}\left(1^2 + \underbrace{\overline{\omega}}_{=\omega^2} + \underbrace{\overline{\omega^2}}_{=\omega}\right) = 0$$



Work the rest out

$$\boxed{x^3 - 1} \checkmark$$

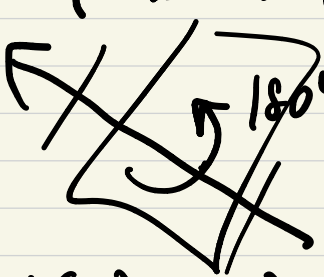
$$
\begin{array}{c|cccc}
A_4 & id & (12)(34) & (123) & (132) \\
 & 1 & 3 & 4 & 4 \\
\hline
\chi_{tr} & 1 & 1 & 1 & 1 \\
\chi_\omega & 1 & 1 & \omega & \omega^2 \\
\chi_{\omega^2} & 1 & 1 & \omega^2 & \omega \\
Tet & 3 & -1 & 0 & 0
\end{array}
$$

$A_4/K_4 = C_3$

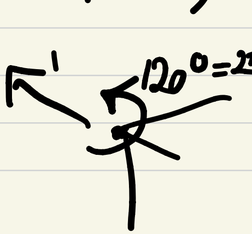
$= \{K_4, (123)K_4, (132)K_4\}$

$id, \quad x, \quad x^2$

$\boxed{\mathbb{C}^3}$ $\begin{bmatrix} 1 & 0 \\ 0 & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix}$



$1 + (-1) + (-1)$



1

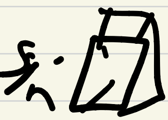
$\cos(2\pi/3) = $

$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

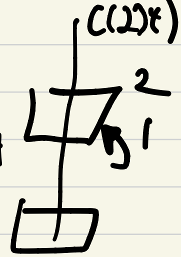
$= -1$

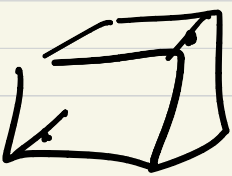
S_4

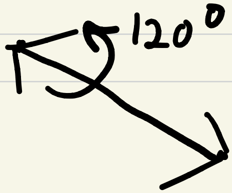
$\left[S_4 / K_4 = S_3 \right]$

	id	c(12)	c(12)(34)	c(123)	c(1234)
	1	6	3	8	6
χ_{id}	1	1	1	1	1
χ_{sgn}	1	−1	1	1	−1
Δ	2	0	2	−1	0
	3	−1	−1	0	1
	3	1	−1	0	−1

$(1^2 + 1^2 + 4 + 9 + \_\_ = 24)$ $\begin{bmatrix} 1 & & \\ & 0 & -1 \\ & 1 & 0 \end{bmatrix}^3$







c(123)

Fun Fact:

If $\chi : G \to \mathbb{C}^*$ is a one-dim'l ch.

and $\rho : G \to \mathrm{Aut}(V)$ is

is a rep, then

$$\chi \cdot \rho\,(g) = \widetilde{\chi(g)} \cdot \rho(g)$$

is a representation!

Rmk: $\chi_\rho = \chi_{\chi\cdot\rho} \iff \rho \cong \chi\cdot\rho$

Challenge! $\underline{\underline{A_5}}$

and S_5

$$y = bx + cy$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_1 \quad x = \underline{ax+cy}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \leftrightarrow x$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \leftrightarrow y$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \underline{\underline{x^2}} = (ax+cy)^2$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} xy = (ax+cy)(bx+dy)$$

y^2

$\mathbb{C}^1$ trivial

$\mathbb{C}^2$ $\quad x, y$ $\leftarrow$ $SL_2(\mathbb{C})$

$\mathbb{C}^3 = \mathrm{Sym}^2(\mathbb{C}^2)$ $\quad x^2, xy, y^2$

$\vdots$

$\mathbb{C}^n = \mathrm{Sym}^n(\mathbb{C}^2)$ $\quad \boxed{x^n, \ldots, y^n}$

Fact: These are <u>all</u> the <u>irreps</u> of $SL(2, \mathbb{C})$

$$\boxed{\mathbb{C}[x,y]} = \boxed{\bigoplus \mathbb{C}[x,y]_d}$$

$SL_2(\mathbb{C})$ $\langle x, y\rangle$ $\qquad \langle x^d, \ldots, y^d\rangle$

$$\boxed{\begin{array}{c}\mathbb{C}[x,y] \\ \cup \\ \left(\text{irrep of } SL_2(\mathbb{C})\right)\end{array}}$$

analogue of $\underline{\mathbb{C}[G]}$ (for $SL(2,\mathbb{C})$)

$\mathcal{U}_n$ $\cap$ $\mathbb{C}[G]$

(reg repn of G)

$$\left(\mathbb{C}[G], \rho\right) \text{ } \underline{\text{reg.}}$$

$\boxed{\text{\underline{Why} are } \mathbb{C}[x,y]_d \text{ irreducible reps of } SL(2,\mathbb{C}) \text{ and why are these \underline{all} of the irred. reps. of } SL(2,\mathbb{C})\text{?}}$