4800-16

Fix a group G and F field

A representation of G on a vector space V is a morphism

$$\rho : G \longrightarrow \mathrm{Aut}(V).$$

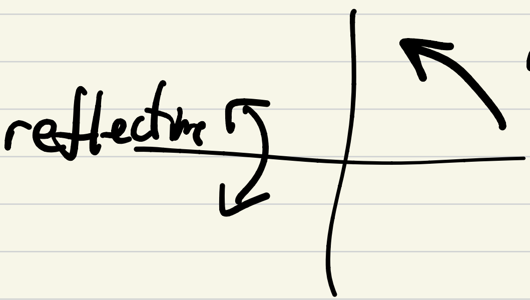
$$\rho(g) : V \xrightarrow{\sim} V$$

Rmk: One usually writes $g \cdot v$ for $\rho(g)(v)$

$\mathbb{R}$-Representations of S_3

Two-dimensional $(\dim V = 2)$

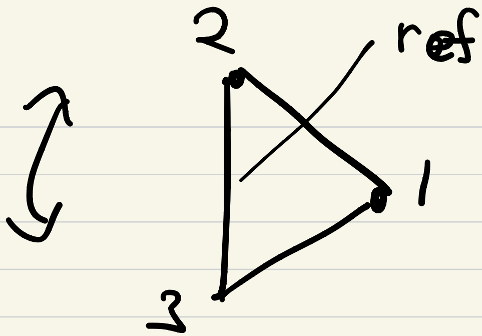
(a) Symmetries of $\triangle$



$$\rho(1\,2\,3) = \text{rotation by } 2\pi/3$$

$$= \begin{bmatrix} \frac{1}{2} & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$\rho(1\,3\,2) = \text{rotation by } 4\pi/3$$



$$\rho(2\,3) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\rho(1\,2) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

<u>Helpful</u> <u>Hint</u>: If

<u>G</u> is generated by elements

$\underline{g_1}, \dots, \underline{g_n}$ with <u>relations</u>, then

to specify a representation need

only to specify

$$\rho(s_1) = A_1$$
$$\rho(s_2) = A_2$$
$$\vdots$$
$$\rho(s_n) = A_n$$

and check the relations on A_i

<u>Example</u>: $\overset{= D_6}{S_3}$ is generated by $g_1 = \underline{(1\ 2)}$ and $g_2 = \underline{(2\ 3)}$

with relations:

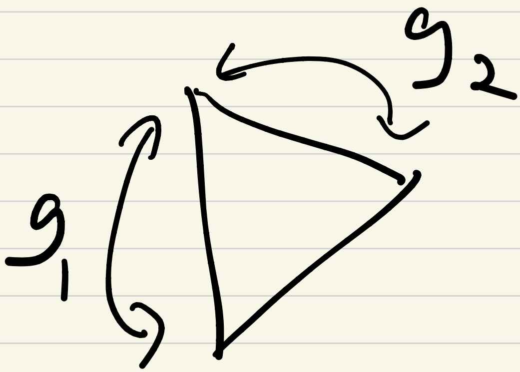
$$g_1^2 = 1,\ g_2^2 = 1,\ (g_1 g_2)^3 = 1$$

$$\left\{\begin{matrix} 1 & x & x^2 \\ y & xy & x^2 y \end{matrix}\right\}$$

$\underline{g_2}$ $\underline{g_1}$

$g_1 g_2 = x$

$x^3 = 1$



$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g_2 = \begin{pmatrix} \frac{1}{2} & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2} & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_1 g_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & +\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$ rotation by $4\pi/3$

<u>Another</u> <u>representation of S_3</u>

$\rho(12) =$
$$\rho(g_1) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = A_1$$

$\rho(23) =$
$$\rho(g_2) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = A_2$$

<u>Check</u>: $A_1^2 = 1_2 = A_2^2 \quad (A_1 A_2)^3 = 1_2$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark$$

$$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = B$$

$$B^3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark$$

Example 2. $\mathbb{R}^1$

Two one-dim'l representations

- Trivial: $g \cdot r = r$

$$\rho(g) = 1 \quad \forall g$$

- Sgn $\sigma \cdot r = \mathrm{sgn}(\sigma) \cdot r$

$$\mathrm{sgn}: S_3 \longrightarrow \underline{\{\pm 1\}}$$

symmetries of $\mathbb{R}$

$\underline{G\text{-Rep}_F}$ category of representations of G on $\underline{F\text{-vector spaces}}$

<u>objects</u>: $\rho : G \to \mathrm{Aut}(V)$

$\rightsquigarrow (V, \rho)$

<u>morphisms</u>: given $\rho_1 : G \to \mathrm{Aut}(V_1)$

$\rho_2 : G \to \mathrm{Aut}(V_2)$,

then a morphism is a G-linear map $f : V_1 \longrightarrow V_2$; $f(g \cdot v) = g f(v)$

$v \in V_1$

$$f(\underbrace{gv}_{V_1}) = g \cdot \underbrace{f(v)}_{V_2}$$

$$\forall g \in G,\ v \in V_1$$

$$f(\underline{\rho_1(g)} \cdot v) = \rho_2(\underline{g}) \cdot \underline{f(v)}$$

$$\rho(g) : \underset{\underset{v}{\cup}}{V} \longrightarrow V$$

Examples:

ρ_1 = trivial rep. on $\mathbb{R}^1$

ρ_2 = sgn rep. on $\mathbb{R}^1$

$f = id : (\mathbb{R}^1, \rho_1) \longrightarrow (\mathbb{R}^1, \rho_2)$

is not G-linear.

$f(r) = r$

$f((1\,2)\cdot r) = f(r) = r$

$(1\,2)\cdot f(r) = sgn(1\,2)\cdot r = -r$

$$(f_1 \circ f_2)(gv)$$

$$= f_1(g \cdot f_2(v))$$

$$= g(f_1 \circ f_2)(v) \quad \checkmark$$

<u>G-linear maps compose.</u>

$$(V, \rho) \xrightarrow{id} (V, \rho)$$

$id: V \to V$ gives

Example: Suppose

(V, ρ) is a G-Rep,

and $U \subset V$ is a subspace

such that

$$g \cdot u \in U \quad \forall g \in G, u \in U.$$

Then U is an invariant subspace of V, and

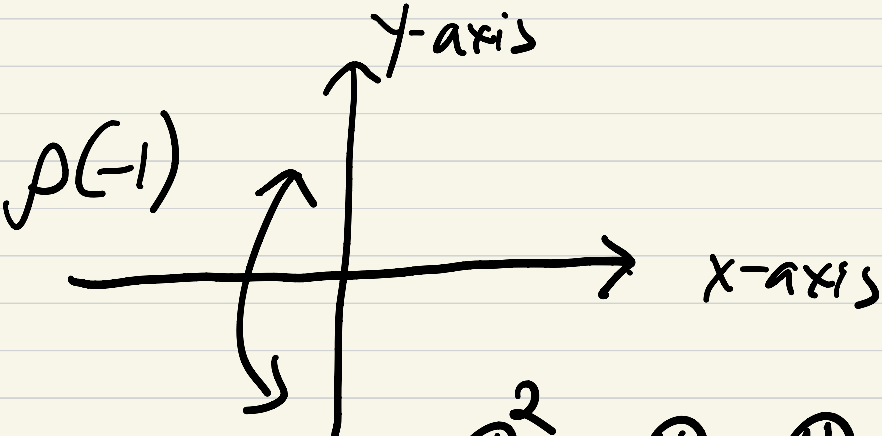
(U, ρ) is a G-Rep.

Example: $G = \{\pm 1\}$

$$\rho : G \longrightarrow \mathrm{Aut}(\mathbb{R}^2)$$

$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho(-1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

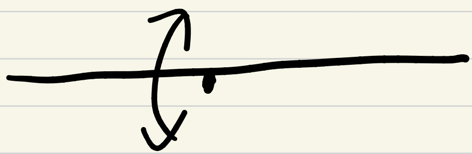


$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

Two invariant subspaces:

$$\mathbb{R}\times 0 \subset \mathbb{R}\times\mathbb{R} \quad \text{x-axis}$$

$$0\times\mathbb{R} \subset \mathbb{R}\times\mathbb{R} \quad \text{y-axis}$$



$(\mathbb{R}\times 0, \rho_{\mathbb{R}})$ trivial

$\rho : \{\pm 1\} \to \mathbb{R}\times 0$ is trivial.

$(0\times\mathbb{R}, \rho)$

$\rho(1)v = v$

$\rho(-1)v = -v$

Permutation rep: of S_n

$$\rho : S_n \longrightarrow \mathrm{Aut}(\mathbb{R}^n)$$

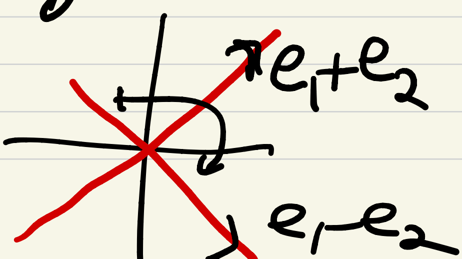
$$\rho(\sigma) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$ is defined by:

$$\rho(\sigma)(e_i) = e_{\sigma(i)}$$

S_2:

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho(1\,2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

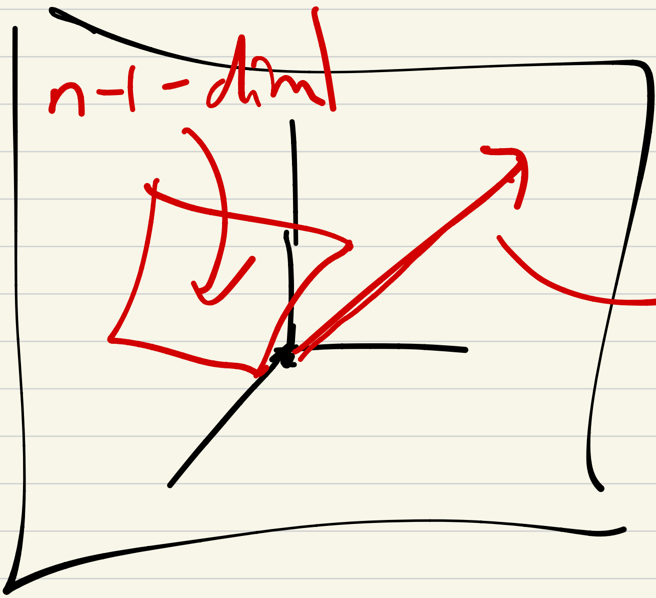


$$\rho(1\,2)(e_1+e_2) = e_2+e_1$$

$$\rho(1\,2)(e_1-e_2) = e_2-e_1 = -(e_1-e_2)$$

· The permutation rep of S_n has **two** invariant **subspaces**:

$\mathbb{R}^n$



$v = e_1 + e_2 + \dots + e_n$

$\rho(\sigma)(v) = v$. Fixed!

one-dim'l subspace (U_1, trivial)

$$U_{n-1} = \langle \underline{e_1 - e_2}, e_2 - e_3, \ldots, e_{n-1} - e_n \rangle$$

$$\underline{\rho(\sigma)} (e_i - e_{i+1}) = e_{\sigma(i)} - e_{\sigma(i+1)}$$

$$= e_j - e_k$$

$$= (e_j - e_{j+1}) + (e_{j+1} - e_{j+2}) + \cdots$$

Explicitly for S_3

$$U_2 = \langle \underbrace{e_1 - e_2}_{v_1}, \underbrace{e_2 - e_3}_{v_2} \rangle \leq \mathbb{R}^3$$

$=$

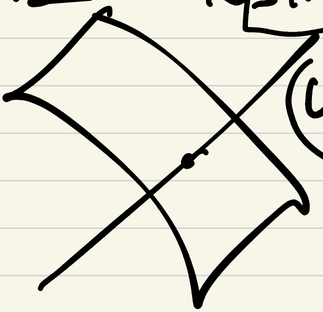
$$\rho(1\ 2)(v_1) = e_2 - e_1 = -v_1$$

$$\rho(1\ 2)(v_2) = e_1 - e_3 = v_1 + v_2$$

$$\rho(\underline{1\ 2}) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$ matrix!

$$\rho(\underline{2\ 3}) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

(U_2, ρ_R) $\boxed{\text{standard rep. of } S_3}$ $\mathbb{R}^3$



$(U_1, \text{trivial})$

$e_1+e_2+e_3$

$(U_1 = \mathbb{R}^1, \text{trivial})$

Def: A representation (V, ρ) is *irreducible* if the only invariant subspaces of V are $\langle 0 \rangle, V$.

Example:

- All one-dim'l reps. of a group are irreducible.

- One-dim'l complex repns

- $\chi : \underline{G} \to \underline{\mathbb{C}}^* = \mathrm{Aut}(\mathbb{C}^1)$.

are called characters.

Eg. Let $C_n = \{1, x, \dots, x^n = 1\}$ cyclic gp.

Then $\chi_m : C_n \to \mathbb{C}^*$ are the characters $\chi_m(x) = \zeta^m$; $\zeta = e^{2\pi i/n}$.

$\underline{C_3}$:

$\chi_1(x)$ $\qquad$ $2\pi/3$

$\chi_2(x)$ $\qquad$ $4\pi/3$

$\chi_3(x)$ $\qquad$ $\underline{id}$

$$\chi_1(x) = \zeta \;,\; \chi_1(x) = \zeta^2 \ldots$$

$$\chi_2(x) = \zeta^2 \;,\; \chi_2(x^2) = \zeta^4 \;,\; \ldots$$