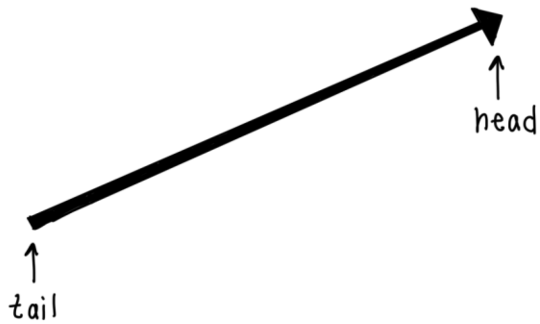
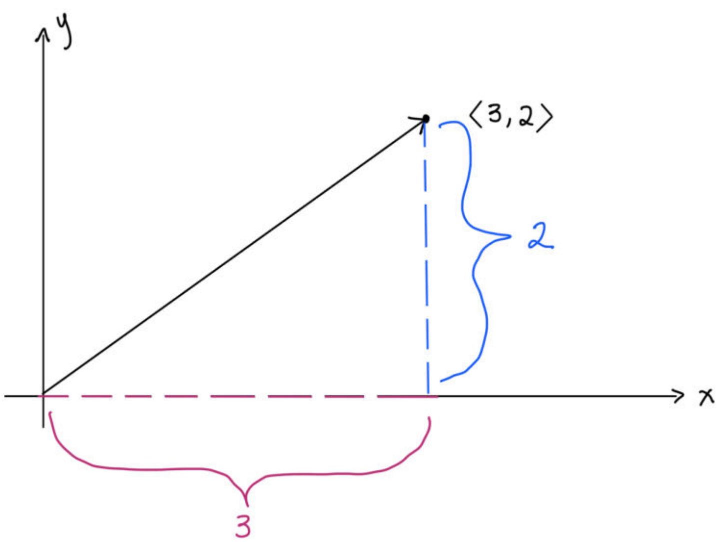
Scribe Notes: Vectors and Products

Definition: Vectors are objects with both direction and magnitude (length). Vectors are drawn as arrows with a tail and head.



Example: Draw vector <3,2> on the plane.



To draw this vector, we need to think about its components. The "3" tells us that the vector has a length of three in the x-direction and "2" represents a length of two in the y-direction.

To calculate the magnitude of any vector, we calculate the distance formula:

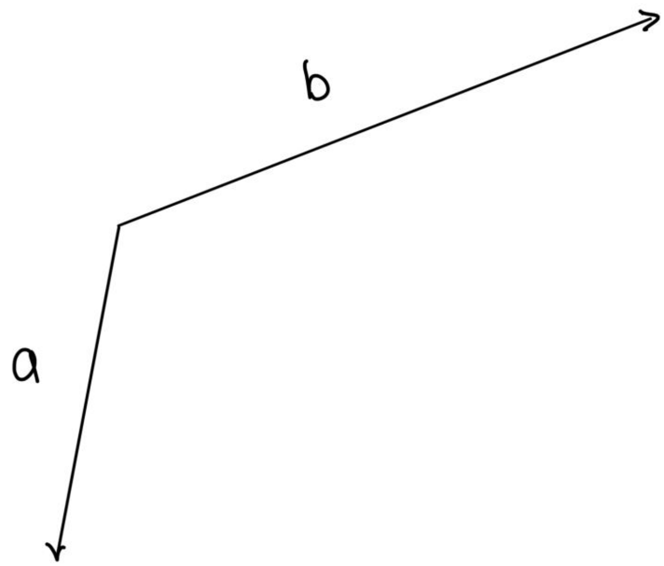
$$||V|| = \sqrt{x^2 + y^2}$$

This stems directly from the Pythagorean Theorem.

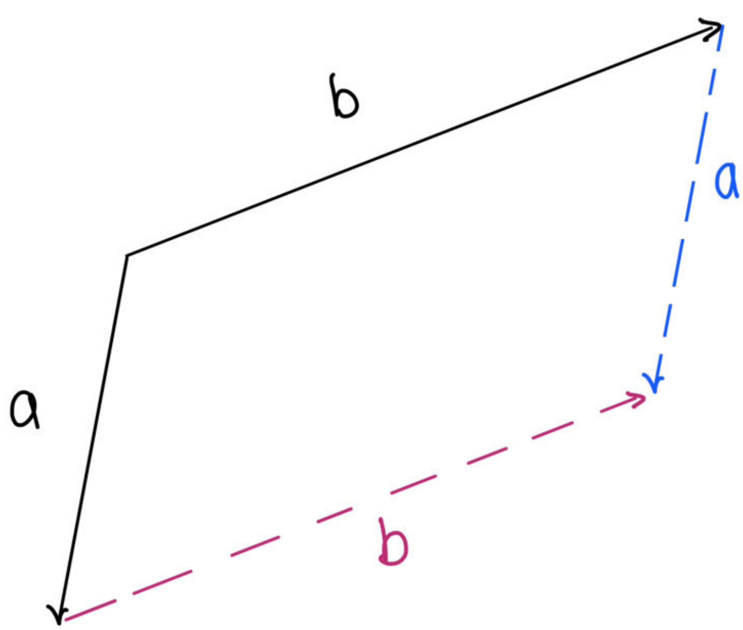
Ex. The magnitude of the vector <3,2> above is $\sqrt{3^2 + 2^2} = \sqrt{9+4} = \sqrt{13}$.

Addition of Vectors:

Say we want to add vectors a and b as pictured below.



Vector addition is done "tail to head". Since vectors are commutative over addition, we can move vector a so its tail lines up at the head of vector b. Doing so creates a parallelogram out of vectors a and b.



The Zero Vector:

The zero vector is an interesting case because its magnitude is well defined (which is zero), but its direction is not. If we cannot draw a vector of length zero, its direction could be nowhere or everywhere!

Dot Product:

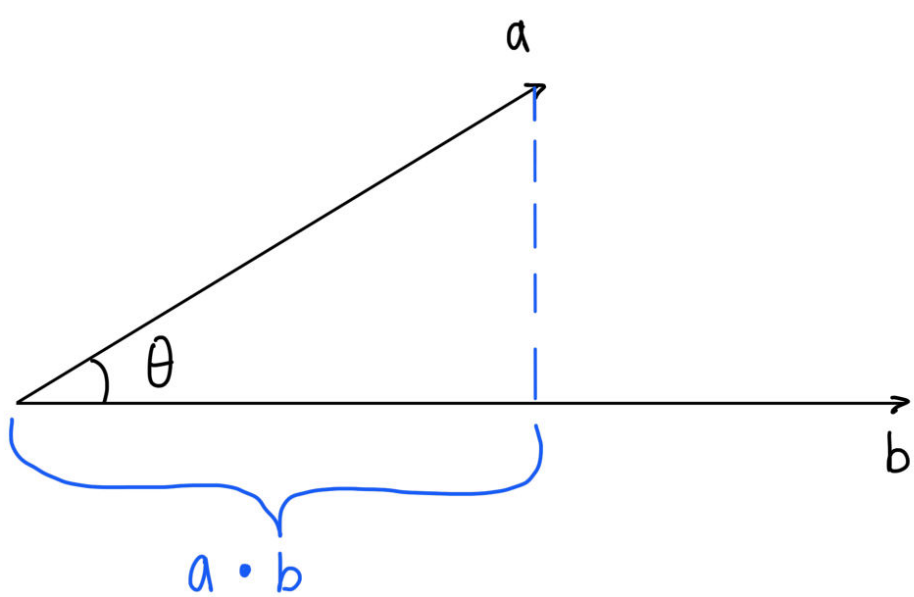
The dot product of two vectors is defined as

$a \cdot b = a_1b_1 + a_2b_2 + \ldots + a_nb_n = \sum_{i=1}^{n} a_ib_i$ where $a = < a_1, a_2, \ldots, a_n >$ and $b = < b_1, b_2, \ldots, b_n >$.

Consequently,

$$a \cdot a = a_1a_1 + a_2a_2 + \ldots + a_na_n = ||a||^2$$

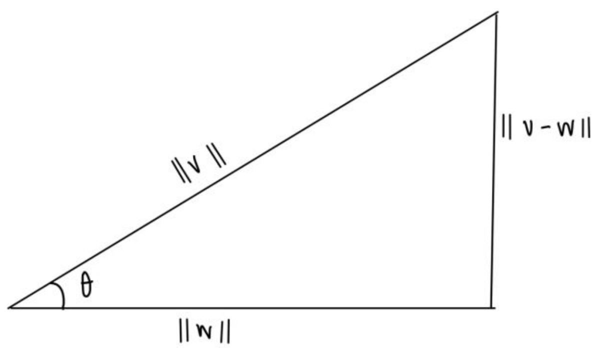
Geometrically, the dot product is the projection of vector a onto vector b.



Then $a \cdot b$ is represented as

$$a \cdot b = ||a||\,||b||\,cos\theta$$

Proof Using Law of Cosines: $a \cdot b = ||a||\ ||b||\ cos\theta$

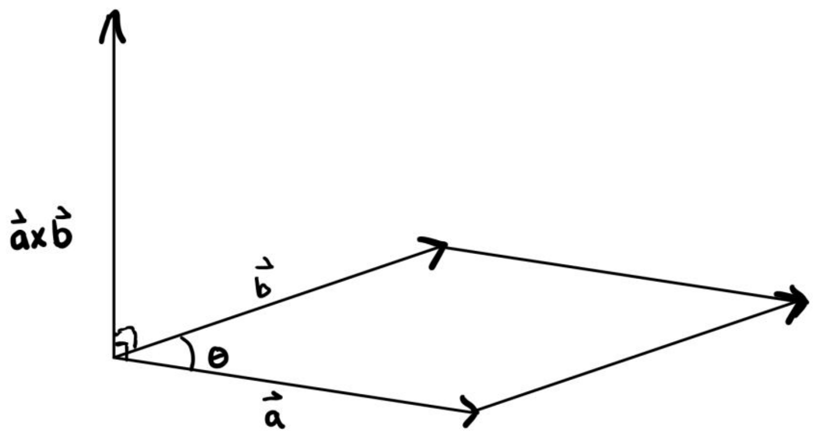


- By the law of cosines:
  - $||v - w||^2 = ||w||^2 + ||v||^2 - 2||v||\ ||w||cos\theta$
- Also,
  - $||v - w||^2 = (v - w) \cdot (v - w)$
  - $||v - w||^2 = v \cdot v - 2(v \cdot w) + w \cdot w$
  - $||v - w||^2 = ||v||^2 + ||w||^2 - 2(v \cdot w)$
- Setting these two equations equal to each other- $||w||^2 + ||v||^2 - 2||v||\ ||w||cos\theta = ||v||^2 + ||w||^2 - 2(v \cdot w)$
- $-2||v||||w||cos\theta = -2(v \cdot w)$
- $||v||||w||cos\theta = (v \cdot w)$
- $v \cdot w = ||v||\ ||w||cos\theta$

The take-away from this formula is that given two vectors, we can find the angle between them.

Cross Product

The cross product of two vectors, in R^3, is an operation that finds a vector orthogonal to both of our initial vectors. The resulting vector, c, is defined to be $a \times b = \|a\|\|b\|(sin\theta)(n)$, where n is



the unit vector that is perpendicular to both vector a and b. We can find the magnitude of our new vector c without knowing c. $\|a \times b\|$ is equal to the area of the parallelogram created by vectors a and b. So $\|a \times b\| = \|b\|sin\theta\|a\|$. We can see this when we look back to our first equation for the dot product; since n is our unit vector in the direction of c, and $\|c\| = \|b\|sin\theta\|a\|$. Our magnitude scales the unit vector to give us c.

Cross Product Hand Rule

In order to help students visualize where $a \times b$ will be in relation to a and b, we teach them the cross product hand rule; using your right hand, point your fingers in the direction of your first vector, a. Then, hold your hand out as if you were about to shake someone's hand. Let your fingers point in the direction of your first vector a. Then, curl your fingers to point in the direction of your second vector b. This may require you to move your hand. Then, whatever direction your thumb is pointing in will be the direction of $a \times b$.

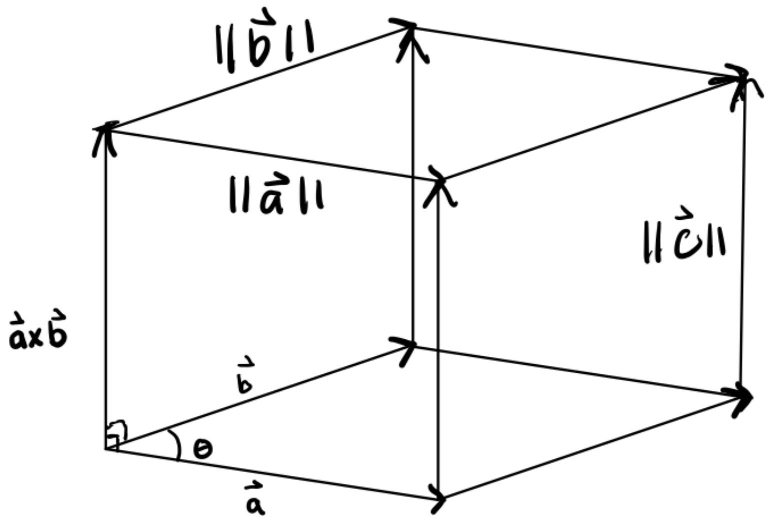
Cross Product as a Matrix

We can also look at the cross product as the determinant of a matrix. Let $a =< a_x, a_y, a_z >$, $b =< b_x, b_y, b_z >$, and $c =< c_x,\ c_y, c_z >$. Then

$$\vec{a} \times \vec{b} = \det \begin{pmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}$$

We can write this as $a \times b = (a_y b_z - a_z b_y)i - (a_x b_z - \ a_z b_x)j + (a_x b_y - a_y b_x)k$. So $c =< (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x >$.

Using this other definition of the cross product, we find $||a \times c|| = 0$ and $||c \times c|| = ||c||^2$. Thinking back to our earlier definition of the cross product and $||c||$, we know $||c||$ equals the area of the parallelogram with side lengths $||a||$ and $||b||$. So $||c||^2$ will be the volume of a shape with a length of $||a||$, width of $||b||$, and height of $||c||$.



As we have seen from our definitions, the cross product only exists in R^3, unlike the dot product. We can also see that the dot product is not associative. Given $i =< 1,0,0 >$, $j =< 0,1,0 >$, and $k =< 0,0,1 >$, $i \times (i \times j) = i \times k = -j$ while $(i \times i) \times j = 0 \times j = 0$.

Applications of the Cross Product

The cross product is used to help us understand electricity and magnetism. William Rowan Hamilton had the idea that it time could be added as a dimension, and as such there there would be an equation $a + bi + cj + dk$, where $a, b, c, d \epsilon R$. And with this new dimension, $i \times i = -1$, $j \times j = -1$, $k \times k = -1$, thus fixing the associativity problem. This is called a Hamiltonian, and acts like a complex number. Hamiltonians are used in Maxwell's equation.